Geometry of Radon measures via Hölder parameterizations

Matthew Badger

Department of Mathematics
University of Connecticut

Geometric Measure Theory
Warwick, United Kingdom
July 10–14, 2017

Research Partially Supported by NSF DMS 1500382, 1650546
Decomposition of Measures

Let $\mu$ be a measure on a measurable space $(X, \mathcal{M})$. Let $\mathcal{N} \subseteq \mathcal{M}$ be a family of measurable sets.

- $\mu$ is carried by $\mathcal{N}$ if there exist countably many sets $\Gamma_i \in \mathcal{N}$ such that $\mu(X \setminus \bigcup_i \Gamma_i) = 0$.
- $\mu$ is singular to $\mathcal{N}$ if $\mu(\Gamma) = 0$ for every $\Gamma \in \mathcal{N}$.

**Exercise (Decomposition Lemma)**

If $\mu$ is $\sigma$-finite, then $\mu$ can be written uniquely as $\mu_{\mathcal{N}} + \mu_{\mathcal{N}}^\perp$, where $\mu_{\mathcal{N}}$ is carried by $\mathcal{N}$ and $\mu_{\mathcal{N}}^\perp$ is singular to $\mathcal{N}$.

- e.g. $\mathcal{N} = \{A \in \mathcal{M} : \nu(A) = 0\}$: $\mu = \sigma + \rho$ where $\sigma \perp \nu$ and $\rho \ll \nu$
- Proof of the Decomposition Theorem is abstract nonsense.

**Identification Problem**: Find measure-theoretic and/or geometric characterizations or constructions of $\mu_{\mathcal{N}}$ and $\mu_{\mathcal{N}}^\perp$?
PSA: Don’t Think About Support

Three Measures. Let $a_i > 0$ be weights with $\sum_{i=1}^{\infty} a_i = 1$. Let $\{x_i : i \geq 1\}$, $\{\ell_i : i \geq 1\}$, $\{S_i : i \geq 1\}$ be a dense set of points, unit line segments, unit squares in the plane.

\[
\begin{align*}
\mu_0 &= \sum_{i=1}^{\infty} a_i \delta_{x_i} \\
\mu_1 &= \sum_{i=1}^{\infty} a_i L^1 \downarrow \ell_i \\
\mu_2 &= \sum_{i=1}^{\infty} a_i L^2 \downarrow S_i
\end{align*}
\]

- $\mu_0$, $\mu_1$, $\mu_2$ are probability measures on $\mathbb{R}^2$
- spt $\mu$ is smallest closed set carrying $\mu$; spt $\mu_0 = \text{spt } \mu_1 = \text{spt } \mu_2 = \mathbb{R}^2$
- $\mu_i$ is carried by $i$-dimensional sets (points, lines, squares)
- The support of a measure is a rough approximation that hides underlying structure of a measure
Rectifiable Measures: Identification Problem Solved for Absolute Continuous Measures

Let $1 \leq m \leq n - 1$ integers. A Radon measure $\mu$ on $\mathbb{R}^n$ is \textit{m-rectifiable} if $\mu$ is carried by images of Lipschitz maps $[0, 1]^m \to \mathbb{R}^n$. $\mu$ is \textbf{purely m-unrectifiable} if $\mu$ is singular to Lipschitz images of $[0, 1]^m$.

\textbf{Theorem (Azzam, Mattila, Preiss, Tolsa, Toro)}

Assume that $\mu \ll \mathcal{H}^m$ ($\iff \lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r^m} < \infty \mu$-a.e.) \textit{TFAE}:

1. $\mu$ is $m$-rectifiable
2. $\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r^m} > 0$, $\text{Tan}(\mu, x) \subseteq \{c\mathcal{H}^m \subseteq V : V \in G(n, m)\} \mu$-a.e.
3. $\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r^m} > 0$ $\mu$-a.e.
4. $\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r^m} > 0$, $\lim_{r \downarrow 0} \left(\frac{\mu(B(x, r))}{r^m} - \frac{\mu(B(x, 2r))}{(2r)^m}\right) = 0 \mu$-a.e.
5. $\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r^m} > 0$, $\int_0^1 \beta_2(\mu, B(x, r))^{2/3} \frac{dr}{r} < \infty \mu$-a.e., where $\beta_2(\mu, B(x, r))$ records "flatness" of $\mu$ in $B(x, r)$

Earlier contributions: Besicovitch, Federer, Marstrand, Morse, Randolph
The study of rectifiability is not done because...

Theorem (Garnett-Killip-Schul 2010)

There exist Radon measures $\mu$ on $\mathbb{R}^2$ with $\text{spt} \mu = \mathbb{R}^2$ such that $\mu$ is 1-rectifiable, $\mu \perp \mathcal{H}^1$, and $\mu$ is doubling ($\mu(B(x, 2r)) \lesssim \mu(B(x, r))$).

$\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r} = \infty \text{ } \mu\text{-a.e.}$

$\int_0^1 \left( \frac{\mu(B(x, r))}{r} \right)^{-1} \frac{dr}{r} < \infty \text{ } \mu\text{-a.e.}$

(see B-Schul 2016)

$\mu(\Gamma) = 0$ whenever $\Gamma = f([0, 1])$ and $f : [0, 1] \rightarrow \mathbb{R}^2$ is bi-Lipschitz

Nevertheless there exist Lipschitz maps $f_i : [0, 1] \rightarrow \mathbb{R}^2$ such that

$\mu \left( \mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} f_i([0, 1]) \right) = 0$
The study of rectifiability is not done because...

**Theorem (Garnett-Killip-Schul 2010)**

*There exist Radon measures $\mu$ on $\mathbb{R}^2$ with $\text{spt} \mu = \mathbb{R}^2$ such that $\mu$ is 1-rectifiable, $\mu \perp \mathcal{H}^1$, and $\mu$ is doubling ($\mu(B(x, 2r)) \lesssim \mu(B(x, r))$).*

- $\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r} = \infty \mu$-a.e.
- $\int_0^1 \left( \frac{\mu(B(x, r))}{r} \right)^{-1} \frac{dr}{r} < \infty \mu$-a.e.

(see B-Schul 2016)

- $\mu(\Gamma) = 0$ whenever $\Gamma = f([0, 1])$ and $f : [0, 1] \to \mathbb{R}^2$ is bi-Lipschitz
- Nevertheless there exist Lipschitz maps $f_i : [0, 1] \to \mathbb{R}^2$ such that

$$
\mu \left( \mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} f_i([0, 1]) \right) = 0
$$
The study of rectifiability is not done because...

**Theorem (Garnett-Killip-Schul 2010)**

There exist Radon measures \( \mu \) on \( \mathbb{R}^2 \) with \( \text{spt} \mu = \mathbb{R}^2 \) such that \( \mu \) is 1-rectifiable, \( \mu \perp \mathcal{H}^1 \), and \( \mu \) is doubling (\( \mu(B(x, 2r)) \lesssim \mu(B(x, r)) \)).

\[ \lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r} = \infty \mu\text{-a.e.} \]

\[ \int_0^1 \left( \frac{\mu(B(x, r))}{r} \right)^{-1} \frac{dr}{r} < \infty \mu\text{-a.e.} \]

(see B-Schul 2016)

\[ \mu(\Gamma) = 0 \] whenever \( \Gamma = f([0, 1]) \) and \( f : [0, 1] \to \mathbb{R}^2 \) is bi-Lipschitz

Nevertheless there exist Lipschitz maps \( f_i : [0, 1] \to \mathbb{R}^2 \) such that

\[ \mu \left( \mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} f_i([0, 1]) \right) = 0 \]
The study of rectifiability is not done because...

**Theorem (Garnett-Killip-Schul 2010)**

There exist Radon measures $\mu$ on $\mathbb{R}^2$ with $\text{spt} \, \mu = \mathbb{R}^2$ such that $\mu$ is 1-rectifiable, $\mu \perp \mathcal{H}^1$, and $\mu$ is doubling ($\mu(B(x, 2r)) \lesssim \mu(B(x, r))$).

- $\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r} = \infty \ \mu$-a.e.
- $\int_0^1 \left( \frac{\mu(B(x, r))}{r} \right)^{-1} \frac{dr}{r} < \infty \ \mu$-a.e.

(see B-Schul 2016)

- $\mu(\Gamma) = 0$ whenever $\Gamma = f([0, 1])$ and $f : [0, 1] \to \mathbb{R}^2$ is bi-Lipschitz
- Nevertheless there exist Lipschitz maps $f_i : [0, 1] \to \mathbb{R}^2$ such that
  $$\mu \left( \mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} f_i([0, 1]) \right) = 0$$
The study of rectifiability is not done because...

Theorem (Garnett-Killip-Schul 2010)

There exist Radon measures $\mu$ on $\mathbb{R}^2$ with $\text{spt} \mu = \mathbb{R}^2$ such that $\mu$ is 1-rectifiable, $\mu \perp \mathcal{H}^1$, and $\mu$ is doubling ($\mu(B(x, 2r)) \lesssim \mu(B(x, r))$).

- $\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r} = \infty$ $\mu$-a.e.
- $\int_0^1 \left( \frac{\mu(B(x, r))}{r} \right)^{-1} \frac{dr}{r} < \infty$ $\mu$-a.e.

(see B-Schul 2016)

- $\mu(\Gamma) = 0$ whenever $\Gamma = f([0, 1])$ and $f : [0, 1] \to \mathbb{R}^2$ is bi-Lipschitz
- Nevertheless there exist Lipschitz maps $f_i : [0, 1] \to \mathbb{R}^2$ such that $\mu \left( \mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} f_i([0, 1]) \right) = 0$
The study of rectifiability is not done because...

**Theorem (Garnett-Killip-Schul 2010)**

There exist Radon measures $\mu$ on $\mathbb{R}^2$ with $\text{spt} \mu = \mathbb{R}^2$ such that $\mu$ is 1-rectifiable, $\mu \perp \mathcal{H}^1$, and $\mu$ is doubling ($\mu(B(x, 2r)) \lesssim \mu(B(x, r))$).

- $\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r} = \infty \; \mu$-a.e.
- $\int_0^1 \left( \frac{\mu(B(x, r))}{r} \right)^{-1} \frac{dr}{r} < \infty \; \mu$-a.e.

(see B-Schul 2016)

- $\mu(\Gamma) = 0$ whenever $\Gamma = f([0, 1])$ and $f : [0, 1] \to \mathbb{R}^2$ is bi-Lipschitz

- Nevertheless there exist Lipschitz maps $f_i : [0, 1] \to \mathbb{R}^2$ such that

$$\mu \left( \mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} f_i([0, 1]) \right) = 0$$
Identification Problem Solved for 1-Rectifiable Measures

Let $1 \leq m \leq n - 1$ integers. A Radon measure $\mu$ on $\mathbb{R}^n$ is **1-rectifiable** if $\mu$ is carried by rectifiable curves (images of Lipschitz maps $[0, 1] \to \mathbb{R}^n$). 

$\mu$ is **purely 1-unrectifiable** if $\mu$ is singular to rectifiable curves.

**Theorem (B, Schul 2017)**

Assume that $\mu$ is a Radon measure on $\mathbb{R}^n$. TFAE:

1. $\mu$ is 1-rectifiable
2. $\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r} > 0$ $\mu$-a.e. and

$$\sum_{Q \in \Delta} \beta_2^*(\mu, 3000Q)^2 \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x) < \infty \quad \mu\text{-a.e.},$$

where $\beta_2^*(\mu, 3000Q)$ records “flatness” of $\mu$ in large dilate of a dyadic cube “nonhomogeneously” and “anisotropically”

One new ingredient: $L^2$ extension of Jones’ traveling salesman theorem that works with **non-doubling measures**. Also see Martikainen and Orponen.
What about $m$-rectifiable measures for $m \geq 2$?

Recent preprints by Azzam-Schul, Edelen-Naber-Valtorta, Ghinassi based on the Reifenberg algorithm give some partial results, but a characterization of 2-rectifiable Radon measures is currently out of reach.

Missing a good characterization of subsets of Lipschitz images of squares. In fact, even the following basic question is wide open.

**Open:** Find extra metric, geometric, and/or topological conditions which ensure a compact, connected set $K \subseteq \mathbb{R}^n$ with $\mathcal{H}^2(K) < \infty$ is contained in the image of a Lipschitz map $f : [0,1]^2 \rightarrow \mathbb{R}^n$.

**A basic enemy:** Let $C$ be the planar four corner Cantor set of dimension 1. Then

$$K = ([0,1]^2 \times \{0\}) \cup (C \times [0,1]) \subset \mathbb{R}^3$$

is connected, compact, and $0 < \mathcal{H}^2(K) < \infty$, but the subset $K' = C \times [0,1]$ is purely 2-unrectifiable.
For each $s \in [1, n]$, let $\mathcal{N}_s$ denote all $(1/s)$-Hölder curves in $\mathbb{R}^n$, i.e. all images $\Gamma$ of $(1/s)$-Hölder continuous maps $f : [0, 1] \to \mathbb{R}^n$.

**Decomposition:** Every Radon measure $\mu$ on $\mathbb{R}^n$ can be uniquely written as $\mu = \mu_{\mathcal{N}_s} + \mu_{\mathcal{N}_s^\perp}$, where

- $\mu_{\mathcal{N}_s}$ is carried by $(1/s)$-Hölder curves
- $\mu_{\mathcal{N}_s^\perp}$ is singular to $(1/s)$-Hölder curves

**Notes**

- Every measure $\mu$ on $\mathbb{R}^n$ is carried by $(1/n)$-Hölder curves (space-filling curves).
- If $\mu$ is $m$-rectifiable, then $\mu$ is carried by $(1/m)$-Hölder curves.
- A measure $\mu$ is 1-rectifiable iff $\mu$ is carried by 1-Hölder curves.
- Martín and Mattila (1988,1993,2000) studied this concept for measures $\mu$ of the form $\mu = \mathcal{H}^s \llcorner E$, where $0 < \mathcal{H}^s(E) < \infty$. 
Essential Examples

“Rectifiable $s$-sets”

- Let $\Gamma$ be a generalized von Koch curve of Hausdorff dimension $s$. Then there exists a $(1/s)$-Hölder map $[0,1] \rightarrow \Gamma$.
- $\mu = \mathcal{H}^s \perp \Gamma$ is carried by $(1/s)$-Hölder curves

“Purely unrectifiable $s$-sets”

Theorem (Martín and Mattila 1993)

Let $K \subseteq \mathbb{R}^n$ be a self-similar Cantor set of Hausdorff dimension $s$. Then $\mu = \mathcal{H}^s \perp K$ is singular to $(1/s)$-Hölder curves.

- This extends a result of Hutchinson (1981) who showed self-similar Cantor sets of Hausdorff dimension $m$ are purely $m$-unrectifiable.

Open Problem (Identification Problem for $s$-sets)

Let $s \in (1, n)$. Characterize $s$-sets $E \subseteq \mathbb{R}^n$ such that $\mu = \mathcal{H}^s \perp E$ is carried by $(1/s)$-Hölder curves. (This is even open when $s = 2$.)
New Results: Measures with Extreme Lower Densities

Theorem (B-Vellis, arXiv 2017)

Let $\mu$ be a Radon measure on $\mathbb{R}^n$ and let $s \in [1, n)$. Then the measure

$$\mu^s_0 := \mu \ll \left\{ x \in \mathbb{R}^n : \lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r^s} = 0 \right\}$$

is singular to $(1/s)$-Hölder curves, i.e. $\mu^s_0(\Gamma) = 0$ for all $(1/s)$-Hölder curves $\Gamma$.

The measure

$$\mu^s_\infty := \mu \ll \left\{ x \in \mathbb{R}^n : \int_0^1 \left( \frac{\mu(B(x, r))}{r^s} \right)^{-1} \frac{dr}{r} < \infty \text{ and } \lim_{r \downarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty \right\}$$

is carried by $(1/s)$-Hölder curves, i.e. $\mu^s_\infty(\mathbb{R}^n \setminus \bigcup_{i=1}^\infty \Gamma_i) = 0$ for some sequence of $(1/s)$-Hölder curves $\Gamma_i$.

- At each $x$, $\int_0^1 \left( \frac{\mu(B(x, r))}{r^s} \right)^{-1} \frac{dr}{r} < \infty$ implies $\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{r^s} = \infty$.
  
  We might call these points of "rapidly infinite" density

- The case $s = 1$ obtained earlier by B-Schul (2015, 2016).
Measures with Positive Lower and Finite Upper Density

Corollary

Let $\mu$ be a Radon measure on $\mathbb{R}^n$, let $s \in [1, n)$ and $t < s$. Then

$$\mu^t_+ := \mu \ll \left\{ x \in \mathbb{R}^n : 0 < \lim_{r \downarrow 0} \frac{\mu(B(x, r))}{rt} \leq \lim_{r \downarrow 0} \frac{\mu(B(x, r))}{rt} < \infty \right\}$$

is carried by $(1/s)$-Hölder curves. (Proof: $t < s$ implies $\mu^t_+ \ll \mu^s_{\infty}$)

Two Refinements

Theorem (B-Vellis, arXiv 2017)

Let $\mu$ be a Radon measure on $\mathbb{R}^n$, let $s \in [m, n)$ and $t < s$. Then $\mu^t_+$ is carried by images of $(m/s)$-Hölder maps $[0, 1]^m \rightarrow \mathbb{R}^n$.

Theorem (B-Vellis, arXiv 2017)

Let $\mu$ be a Radon measure on $\mathbb{R}^n$ and let $t < 1$. Then $\mu^t_+$ is carried by images of bi-Lipschitz maps $[0, 1] \rightarrow \mathbb{R}^n$. 
Example: $2^n$-corner Cantor sets

Let $K_t \subset [0,1]^n$ be the self-similar $2^n$-corner Cantor set of Hausdorff dimension $t \in (0,n)$. Let $1 \leq m \leq n - 1$ be integers.

- If $t \in [m, n)$, then $\mathcal{H}^t \subset K_t$ is singular to $(m/t)$-Hölder images of $[0,1]^m$ [Martín and Mattila 1993]

- If $t \in [m, n)$, then $\mathcal{H}^t \subset K_t$ is carried by $(m/s)$-Hölder images of $[0,1]^m$ for all $s > t$ [Martín and Mattila 2000] or [B-Vellis]

- If $t \in (0, 1)$, then $\mathcal{H}^t \subset K_t$ is carried by bi-Lipschitz curves [B-Vellis]
Hölder Parameterization of Leaves of Summable Trees

A tree off dyadic cube $T$ is a collection of dyadic cubes with maximal element $Q_0$ such that if $Q \in T$ and $Q \subsetneq Q_0$, then $Q^\uparrow \in T$. A leaf of $T$ is a limit of a sequence sampled from an infinite branch of $T$.

**Theorem (B-Vellis arXiv 2017)**

Let $T$ be a tree of dyadic cubes (or similar tree of sets). If $s \geq 1$ and

$$\sum_{Q \in T} (\text{diam } Q)^s < \infty,$$

then $\mathcal{H}^s(\text{Leaves}(T)) = 0$ and there is a $(1/s)$-Hölder curve $\Gamma$ such that $\text{Leaves}(T) \subseteq \Gamma$.

- When $s = 1$ this was proved by B-Schul (2016) using the special fact that every connected, compact set with finite $\mathcal{H}^1$ measure is a rectifiable curve.
- When $s > 1$, have to construct the Hölder parameterizations by hand.
Hölder and Bi-Lipschitz Parameterization of Sets of “Small” Assouad Dimension

For $E \subseteq \mathbb{R}^n$, let $\dim_A(E)$ denote its **Assouad dimension**

**Theorem (B-Vellis arXiv 2017)**

Let $s \in [m, n)$. If $E \subseteq \mathbb{R}^n$ is a bounded set with $\dim_A(E) < s$, then there is an $(m/s)$-Hölder map $f : [0, 1]^m \to \mathbb{R}^n$ such that $E \subseteq f([0, 1]^m)$.

**Theorem (B-Vellis arXiv 2017)**

If $E \subseteq \mathbb{R}^n$ is a bounded set with $\dim_A(E) < m$ and if the set $E$ is uniformly disconnected in sense of David and Semmes, then there exists a bi-Lipschitz map $f : [0, 1]^m \to \mathbb{R}^n$ such that $E \subseteq f([0, 1]^m)$.

- When $\dim_A(E) < 1$, the set $E$ is always uniformly disconnected.
- Proof of these results is constructive. Borrows ideas from MacManus’ construction of quasicircles passing through uniformly disconnected sets.
Proof of Bi-Lipschitz Parameterization

1. Simple reduction: enough to consider compact sets in the codimension 1 case
2. Use uniform disconnectedness to approximate set by a sequence of manifolds with boundary, $\partial M$ contained in faces of standard grid
3. Construct tree-like surfaces passing through successive approximations:
Takeaways

1. **General Problem in Geometry of Measures:**
   Let $(X, \mathcal{M})$ be a measure space and let $\mathcal{N}$ be a family of measurable sets. Find geometric and/or measure-theoretic characterizations of measures that are
   - carried by $\mathcal{N}$ (rectifiable measures), or
   - singular to $\mathcal{N}$ (purely unrectifiable measures).

   While this problem has been well-studied in $\mathbb{R}^n$ under certain regularity assumptions (absolutely continuous measures), there are many open questions when we drop regularity (Radon measures) or change the space $X$ or choose different sets $\mathcal{N}$.

2. **Non-integral Rectifiability:**
   One candidate for rectifiability in non-integral dimensions based on Hölder continuous images. Some preliminary results have been obtained, but as above there is still more to do!
Thank you