Multiscale Analysis of 1-Rectifiable Measures

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Part I  Rectifiable Measures

Part II  $L^2$ Beta Numbers and Jones Functions

Part III  New Results
General Definition

Let $\mu$ be a Borel measure on $\mathbb{R}^n$ and let $1 \leq m \leq n - 1$. We say that $\mu$ is $m$-rectifiable if there exist countably many

- Lipschitz maps $f_i : [0, 1]^m \to \mathbb{R}^n$

such that

$$\mu \left( \mathbb{R}^n \setminus \bigcup_i f_i([0, 1]^m) \right) = 0.$$ 

(Federer’s terminology: $\mathbb{R}^n$ is countably $(\mu, m)$-rectifiable.)

Examples

- rectifiable curves/surfaces: $\mathcal{H}^m \subseteq f([0, 1]^m)$,
- (countably) rectifiable sets: $\sum_i \mathcal{H}^m \subseteq E_i, \quad E_i \subset f_i([0, 1]^m)$
- Dirac mass $\delta_x$ at $x \in \mathbb{R}^n$
Surprising Example

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure \( \mu \) on \( \mathbb{R}^n \) \((n \geq 2)\) with support \( \mathbb{R}^n \) such that \( \mu \perp \mathcal{H}^1 \), but \( \mu \) is 1-rectifiable.
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Grades of Rectifiable Measures

\{ \text{m-rectifiable measures } \mu \text{ on } \mathbb{R}^n \} \cup \\uplus \\cup \\uplus \\cup \\uplus

\{ \text{m-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ such that } \mu \ll \mathcal{H}^m \} \cup \\uplus

\{ \text{m-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ of the form } \mu = \mathcal{H}^m \llcorner E \}
Absolutely Continuous Rectifiable Measures

The lower and upper (Hausdorff) $m$-density of a measure $\mu$ at $x$:

$$D_m(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B(x, r))}{cmr^m} \quad D^m_m(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu(B(x, r))}{cmr^m}.$$  

Write $D_m(\mu, x)$, the $m$-density of $\mu$ at $x$, if $D_m(\mu, x) = D_m^m(\mu, x)$.

Theorem (Mattila 1975)

Suppose that $E \subset \mathbb{R}^n$ is Borel and $\mu = \mathcal{H}^m \res E$ is locally finite. Then $\mu$ is $m$-rectifiable if and only if $D_m(\mu, x) = 1$ $\mu$-a.e.

Theorem (Preiss 1987)

Suppose that $\mu$ is a locally finite Borel measure on $\mathbb{R}^n$. Then $\mu$ is $m$-rectifiable and $\mu \ll \mathcal{H}^m$ if and only if $0 < D_m(\mu, x) < \infty$ $\mu$-a.e.

There are additional characterizations (tangent measures, etc.)
General Rectifiable Measures

Problem
For all $1 \leq m \leq n - 1$, find necessary and sufficient conditions for a locally finite Borel measure $\mu$ on $\mathbb{R}^n$ to be $m$-rectifiable.

- Do not assume $\mu \ll H^m$.

Theorem (B-Schul)

**Necessary condition** for the case $m = 1$ and $n \geq 2$:

If $\mu$ is 1-rectifiable, then at $\mu$-almost every $x \in \mathbb{R}^n$,

- $\mu \ll B(x, r)$ concentrates mass around a line $\ell_{x,r}$ as $r \to 0$; or
- the density $\mu(B(x, r))/r \to \infty$ sufficiently fast as $r \to 0$. 
Part I Rectifiable Measures

Part II $L^2$ Beta Numbers and Jones Functions

Part III New Results
$L^2$ Beta Numbers

Let $\mu$ be a locally finite Borel measure on $\mathbb{R}^n$ and $Q \subset \mathbb{R}^n$ a cube. Define the $L^2$ beta number $\beta_2^2(\mu, Q) \in [0, 1]$ by

$$
\beta_2^2(\mu, Q) = \inf_{\ell} \int_Q \left( \frac{\text{dist}(x, \ell)}{\text{diam} Q} \right)^2 \frac{d\mu(x)}{\mu(Q)}
$$

where the infimum runs over all lines $\ell$ in $\mathbb{R}^n$. 

\[ \beta_2 = 0 \quad \beta_2 \text{ small} \quad \beta_2 \sim 1 \]
$L^2$ Jones Functions

A collection $\{w(\mu, Q)\}$ of weights $\leadsto$ weighted $L^2$ Jones function:

$$J_2^w(\mu, r, x) = \sum_{\text{side } Q \leq r} \beta_2^2(\mu, 3Q)w(\mu, Q)\chi_Q(x).$$

Two Special Cases

$w(\mu, Q) \equiv 1 \leadsto \text{ordinary } L^2 \text{ Jones function}$

$$J_2(\mu, r, x) = \sum_{\text{side } Q \leq r} \beta_2^2(\mu, 3Q)\chi_Q(x).$$

$w(\mu, Q) \equiv \left(\frac{\mu(Q)}{\text{diam } Q}\right)^{-1} \leadsto \text{density-normalized } L^2 \text{ Jones function}$

$$\tilde{J}_2(\mu, r, x) = \sum_{\text{side } Q \leq r} \beta_2^2(\mu, 3Q)\frac{\text{diam } Q}{\mu(Q)}\chi_Q(x).$$
Ordinary Jones Function and Rectifiable Sets

A Borel measure $\mu$ on $\mathbb{R}^n$ is $m$-Ahlfors regular if $\mu(B(x, r)) \sim r^m$ for all $x$ in the support of $\mu$ and for all $0 < r < r_0(\mu)$.

**Theorem (David-Semmes 1991)**

Suppose $E \subset \mathbb{R}^n$ is closed and $\mu = \mathcal{H}^m \upharpoonright E$ is $m$-AR. Then $\mu$ is uniformly $m$-rectifiable if and only if

$$\int_{B(x_0, r)} J_2(\mu, r, x) d\mu(x) \lesssim r^m \text{ for all } x_0 \in E, \ 0 < r < \text{diam } E.$$  

**Theorem (Pajot 1997)**

Suppose $K \subset \mathbb{R}^n$ is compact and $\mu = \mathcal{H}^m \upharpoonright K$.

- Suppose $\mu$ is $m$-AR. Then $\mu$ is $m$-rectifiable if and only if $J_2(\mu, x) < \infty \ \mu$-a.e.

- Suppose $\mathcal{H}^m(K) < \infty$. Then $\mu$ is $m$-rectifiable if both $D^m(\mu, x) > 0$ and $J_2(\mu, x) < \infty \ \mu$-a.e.
Part I Rectifiable Measures

Part II $L^2$ Beta Numbers and Jones Functions

Part III New Results
Necessary Conditions for 1-Rectifiable Measures

Theorem (B-Schul)

Let $\mu$ be a locally finite Borel measure on $\mathbb{R}^n$.

- If $\mu$ is 1-rectifiable, then

$$\tilde{J}_2(\mu, x) = \sum_{\text{side } Q \leq 1} \beta_2^2(\mu, 3Q) \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x) < \infty \quad \mu\text{-a.e.}$$

- If $\mu$ is 1-rectifiable and $\mu \ll \mathcal{H}^1$, then

$$J_2(\mu, x) = \sum_{\text{side } Q \leq 1} \beta_2^2(\mu, 3Q) \chi_Q(x) < \infty \quad \mu\text{-a.e.}$$

Corollary (B-Schul + Pajot 1997)

Suppose $K \subset \mathbb{R}^n$ is compact and $\mathcal{H}^1(K) < \infty$. Then $\mu = \mathcal{H}^1 \res K$ 1-rectifiable if and only if $D^1(\mu, x) > 0$ and $J_2(\mu, x) < \infty \quad \mu\text{-a.e.}$
Proposition (B-Schul)

Suppose \( \nu(\mathbb{R}^n) < \infty, \Gamma \subset \mathbb{R}^n \) is a rectifiable curve, \( E \subset \Gamma \) is Borel and \( \nu(E \cap B(x, r)) \geq cr \) for all \( x \in E \) and \( 0 < r \leq r_0 \). Then

\[
\int_E \tilde{J}_2(\nu, r_0, x) d\nu(x) \lesssim_{n,c} \mathcal{H}^1(\Gamma) + \nu(\mathbb{R}^n \setminus \Gamma).
\]

\( \tilde{J}_2 \) is defined by

\[
\tilde{J}_2(\nu, r_0, x) = \sum_{\text{side } Q \leq r_0} \beta_2^2(\nu, 3Q) \text{diam } Q \frac{\nu(E \cap Q)}{\nu(Q)}
\]

- Dyadic cubes \( Q \) with \( \nu(E \cap Q) > 0 \) belong to two classes:
  \( \{ \beta_2^2(\nu, 3Q) \lesssim \beta_\Gamma(3Q) \} \) and \( \{ \beta_\Gamma(3Q) \ll \beta_2^2(\nu, 3Q) \} \)

- Sum over first class \( \lesssim_n \mathcal{H}^1(\Gamma) \): Traveling Salesman Theorem for Rectifiable Curves (Jones 1990 in \( \mathbb{R}^2 \), Okikiolu 1992 in \( \mathbb{R}^n \))

- Sum over second class \( \lesssim_{n,c} \nu(\mathbb{R}^n \setminus \Gamma) \): New Estimate!
Future Directions

Problem
For all $1 \leq m \leq n - 1$, find necessary and sufficient conditions for a locally finite Borel measure $\mu$ on $\mathbb{R}^n$ to be $m$-rectifiable.

- Do not assume $\mu \ll \mathcal{H}^m$.

Necessary Conditions
- If $\mu$ is $m$-rectifiable, then $D_m^m(\mu, x) > 0$ $\mu$-a.e.
- If $\mu$ is 1-rectifiable, then $\tilde{J}_2(\mu, x) < \infty$ $\mu$-a.e. (B-Schul)
- What happens for $m$-rectifiable measures, $m \geq 2$?

Sufficient Conditions
- If $D_1^1(\mu, x) > 0$ and $\tilde{J}_2(\mu, x) < \infty$ $\mu$-a.e., is $\mu$ 1-rectifiable?
- Same ? is open for $\mu \ll \mathcal{H}^1$ (but settled for $\mu = \mathcal{H}^1 \restriction K$).