Preview: Structure of Measures

Three Measures. Let $a_i > 0$ be weights with $\sum_{i=1}^{\infty} a_i = 1$.
Let $\{x_i : i \geq 1\}, \{\ell_i : i \geq 1\}, \{S_i : i \geq 1\}$ be a dense set of points, unit line segments, unit squares in the plane.

$\mu_0 = \sum_{i=1}^{\infty} a_i \delta_{x_i}$
$\mu_1 = \sum_{i=1}^{\infty} a_i L^1|_{\ell_i}$
$\mu_2 = \sum_{i=1}^{\infty} a_i L^2|_{S_i}$

- $\mu_0, \mu_1, \mu_2$ are probability measures on $\mathbb{R}^2$
- The support of $\mu$ is the smallest closed set carrying $\mu$;
  $\text{spt } \mu_0 = \text{spt } \mu_1 = \text{spt } \mu_2 = \mathbb{R}^2$
- $\mu_i$ is carried by $i$-dimensional sets (points, lines, squares)
- The support of a measure is a rough approximation that hides the underlying structure of a measure
Part I. Curves

Part II. Subsets of Curves

Part III. Rectifiability of Measures
What is a curve?

A curve $\Gamma \subset \mathbb{R}^n$ is a **continuous image** of $[0, 1]$:

There exists a continuous map $f : [0, 1] \to \mathbb{R}^n$ such that $\Gamma = f([0, 1])$

A continuous map $f$ with $\Gamma = f([0, 1])$ is called a **parameterization** of $\Gamma$

- There are curves which do not have a 1-1 parameterization
- There are curves which have topological dimension $> 1$

A curve $\Gamma$ is **rectifiable** if $\exists f$ with $\sup_{x_0 \leq \cdots \leq x_k} \sum_{j=1}^k |f(x_j) - f(x_{j-1})| < \infty$
When I think of curves...
When is a set a curve?

Theorem (Hahn-Mazurkiewicz)

A nonempty set $\Gamma \subset \mathbb{R}^n$ is a curve if and only if

$\Gamma$ is compact, connected, and locally connected

The proof of the forward direction is an exercise.

The proof of the reverse direction is content of the theorem: must **construct a parameterization** from only topological information.
Examples of sets which are not curves

Theorem (Hahn-Mazurkiewicz)

A nonempty set $\Gamma \subset \mathbb{R}^n$ is not a curve if and only if

$\Gamma$ is not compact or disconnected or not locally connected

Unbounded

Not Closed

Disconnected

Not Locally Connected

a straight line

an open line segment

a Cantor set

a comb
When is a set a rectifiable curve?

Theorem (Ważewski)

Let $\Gamma \subset \mathbb{R}^n$ be nonempty. TFAE:

1. $\Gamma$ is a rectifiable curve (finite total variation)
2. $\Gamma$ is compact, connected, and $\mathcal{H}^1(\Gamma) < \infty$
3. $\Gamma$ is a Lipschitz curve, i.e. there exists a Lipschitz continuous map $f : [0, 1] \to \mathbb{R}^n$ such that $\Gamma = f([0, 1])$

$\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.

$f$ is Lipschitz if $\exists C < \infty$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y$.

The proof of $(1) \Rightarrow (2)$ is an exercise.

The proof of $(3) \Rightarrow (1)$ is trivial.
\( \Gamma \subset \mathbb{R}^n \) is compact, connected, \( \mathcal{H}^1(\Gamma) < \infty \) \( \implies \) \( \Gamma \) is Lipschitz curve

Goal: build a parameterization for the set \( \Gamma \)
\( \Gamma \subset \mathbb{R}^n \) is compact, connected, \( \mathcal{H}^1(\Gamma) < \infty \) \( \implies \) \( \Gamma \) is Lipschitz curve

Step 1: approximate \( \Gamma \) by \( 2^{-k} \)-nets \( V_k, k \geq 1 \)
Proof by Picture

\[ \Gamma \subset \mathbb{R}^n \text{ is compact, connected, } \mathcal{H}^1(\Gamma) < \infty \implies \Gamma \text{ is Lipschitz curve} \]

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Step 2: draw piecewise linear spanning tree \Gamma_k through \mathcal{V}_k
Proof by Picture

$\Gamma \subset \mathbb{R}^n$ is compact, connected, $\mathcal{H}^1(\Gamma) < \infty \implies \Gamma$ is Lipschitz curve

Step 2: draw piecewise linear spanning tree $\Gamma_k$ through $V_k$
\[ \Gamma \subset \mathbb{R}^n \text{ is compact, connected, } \mathcal{H}^1(\Gamma) < \infty \implies \Gamma \text{ is Lipschitz curve} \]

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Step 3: pick a 2-1 tour of edges in the tree \( \Gamma_k \)
Proof by Picture

\[ \Gamma \subset \mathbb{R}^n \text{ is compact, connected, } \mathcal{H}^1(\Gamma) < \infty \implies \Gamma \text{ is Lipschitz curve} \]

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Γ ⊂ \mathbb{R}^n is compact, connected, \mathcal{H}^1(Γ) < ∞ \implies Γ is Lipschitz curve

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Step 3: pick a 2-1 tour of edges in the tree $\Gamma_k$
Proof by Picture

\(\Gamma \subset \mathbb{R}^n\) is compact, connected, \(\mathcal{H}^1(\Gamma) < \infty \implies \Gamma \) is Lipschitz curve

Step 3: pick a 2-1 tour of edges in the tree \(\Gamma_k\)
Proof by Picture

Γ ⊂ \mathbb{R}^n is compact, connected, \mathcal{H}^1(\Gamma) < \infty \implies \Gamma \text{ is Lipschitz curve}

Step 3: pick a 2-1 tour of edges in the tree Γ_k
Proof by Picture

\[ \Gamma \subset \mathbb{R}^n \text{ is compact, connected, } \mathcal{H}^1(\Gamma) < \infty \implies \Gamma \text{ is Lipschitz curve} \]

Step 3: pick a 2-1 tour of edges in the tree \( \Gamma_k \)
Proof by Picture

\[ \Gamma \subset \mathbb{R}^n \text{ is compact, connected, } \mathcal{H}^1(\Gamma) < \infty \iff \Gamma \text{ is Lipschitz curve} \]

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Step 4: tour defines piecewise linear map \( f_k : [0, 1] \rightarrow \Gamma_k \)
\( \Gamma \subset \mathbb{R}^n \) is compact, connected, \( \mathcal{H}^1(\Gamma) < \infty \implies \Gamma \) is Lipschitz curve

Step 5: length of \( i \)-th edge \( \lesssim \mathcal{H}^1(E \cap B(v_i, \frac{1}{4} \cdot 2^{-k})) \)
\[ \Gamma \subset \mathbb{R}^n \text{ is compact, connected, } \mathcal{H}^1(\Gamma) < \infty \implies \Gamma \text{ is Lipschitz curve} \]

Conclusion: \( \text{Lip } f_k \leq 32\mathcal{H}^1(\Gamma) \). Hence \( f_{kj} \implies f : [0, 1] \to \Gamma \text{ Lipschitz} \)
Open Problem #1

Theorem (Ważyewski)

Let $\Gamma \subset \mathbb{R}^n$ be nonempty. TFAE:

1. $\Gamma$ is a rectifiable curve (finite total variation)
2. $\Gamma$ is compact, connected, and $\mathcal{H}^1(\Gamma) < \infty$
3. $\Gamma$ is a Lipschitz curve, i.e. there exists a Lipschitz continuous map $f : [0, 1] \to \mathbb{R}^n$ such that $\Gamma = f([0, 1])$

Generalize Ważyewski’s theorem to higher dimensional curves
Snowflakes and Squares
Snowflakes and Squares

Open Problem (#2)

For each real \( s \in (1, \infty) \), characterize curves \( \Gamma \subset \mathbb{R}^n \) with \( \mathcal{H}^s(\Gamma) < \infty \)

Open Problem (#3)

For each real \( s \in (1, \infty) \), characterize \((1/s)\)-Hölder curves, i.e. sets that can be presented as \( h([0, 1]) \) for some map \( h : [0, 1] \to \mathbb{R}^n \) with

\[
|h(x) - h(y)| \leq C|x - y|^{1/s}
\]

Open Problem (#4)

For each integer \( m \geq 2 \), characterize Lipschitz \( m \)-cubes, i.e. sets that can be presented as \( f([0, 1]^m) \) for some Lipschitz map \( f : [0, 1]^m \to \mathbb{R}^n \).
Obstruction to a Hölder Ważewski Theorem

- Every \((1/s)\)-Hölder curve has \(\mathcal{H}^s(\Gamma) < \infty\)
- There are curves \(\Gamma\) with \(\mathcal{H}^s(\Gamma) < \infty\) that are not \((1/s)\)-Hölder.

Theorem (B, Naples, Vellis 2018)

*For all \(s > 1\), there exists a curve \(\Gamma \subset \mathbb{R}^n\) such that \(\mathcal{H}^s(\Gamma \cap B(x, r)) \sim r^s\), but \(\Gamma\) is not a \((1/s)\)-Hölder curve.*

Idea.

Look at the cylinder \(C \times [0, 1] \subset \mathbb{R}^2\) over the standard “middle thirds” Cantor set \(C \subset \mathbb{R}\). Adjoining the line segment \([0, 1] \times \{0\}\) makes the set connected, but it is not locally connected. Adjoining additional intervals \(I_i \times \{t_j\}\) on a dense set of heights (“rungs”) makes the set locally connected. We call this a **Cantor ladder**.

A modified version of this gives the desired set.
Obstruction to a Hölder Ważewski Theorem

- Every $(1/s)$-Hölder curve has $\mathcal{H}^s(\Gamma) < \infty$
- There are curves $\Gamma$ with $\mathcal{H}^s(\Gamma) < \infty$ that are not $(1/s)$-Hölder.

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For all $s > 1$, there exists a curve $\Gamma \subset \mathbb{R}^n$ such that $\mathcal{H}^s(\Gamma \cap B(x, r)) \sim r^s$, but $\Gamma$ is not a $(1/s)$-Hölder curve.

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A modified version of this gives the desired set.
Sufficient conditions for Hölder curves

Theorem (Remes 1998)

Let $S \subset \mathbb{R}^n$ be a self-similar set satisfying the open set condition. If $S$ is connected, then $S$ is a $(1/s)$-Hölder curve, $s = \dim_H S$.

A set $E \subset \mathbb{R}^n$ is $\varepsilon$-flat if for every $x \in E$ and $0 < r \leq \text{diam } E$, there exists a line $\ell$ such that $\text{dist}(x, \ell) \leq \varepsilon r$ for all $x \in E \cap B(x, r)$.

Theorem (B, Naples, Vellis 2018)

Assume that $E \subset \mathbb{R}^n$ is $\varepsilon$-flat with $\varepsilon \ll 1$. If $E$ is connected, compact, $\mathcal{H}^s(E) < \infty$ and $\mathcal{H}^s(E \cap B(x, r)) \gtrsim r^s$, then $E$ is a $(1/s)$-Hölder curve with a one-to-one parameterization.
Sufficient conditions for Hölder curves

**Theorem (Remes 1998)**

Let $S \subset \mathbb{R}^n$ be a self-similar set satisfying the open set condition. If $S$ is connected, then $S$ is a $(1/s)$-Hölder curve, $s = \dim_H S$.

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Part I. Curves

Part II. Subsets of Curves

Part III. Rectifiability of Measures
Analyst’s Traveling Salesman Problem

Given a bounded set \( E \subset \mathbb{R}^n \) (an infinite list of cities),
decide whether or not \( E \) is a subset of a rectifiable curve.

If so, construct a rectifiable curve \( \Gamma \) containing \( E \) that is
short as possible.

This is solved for

- \( E \) in \( \mathbb{R}^2 \) by P. Jones (1990)
- \( E \) in \( \mathbb{R}^n \) by K. Okikiolu (1992)
- \( E \) in \( \ell_2 \) by R. Schul (2007)
- \( E \) in first Heisenberg group \( \mathbb{H}^1 \) by S. Li and R. Schul (2016)
- \( E \) in Laakso-type spaces by G.C. David and R. Schul (2017)
- \( E \) is Carnot group by V. Chousionis, S. Li, S. Zimmerman (2018):
  necessary condition only
Not contained in a rectifiable curve: a countable compact set with one accumulation point

For each \( k \geq 2 \), choose \( m_k = k^2 \) so that \( \sum_{k=2}^{\infty} m_k^{-1} < \infty \). Arrange squares \( S_k \) with side length \( m_k^{-1} \) so that one side of each square lies on a given line; separate \( S_k \) and \( S_{k+1} \) by distance \( m_k^{-1} \). Let \( V_k \) be collection of \( m_k^2 \) points in \( S_k \) separated by distance at least \( m_k^{-2} \). Let \( E \) be the closure of \( \bigcup_{k=2}^{\infty} V_k \).

Suppose \( \Gamma = f([0, 1]) \supseteq E \) for some \( f \) with \( |x - y| \geq L^{-1}|f(x) - f(y)| \).
To contain \( V_k \), the curve \( \Gamma \) must cross \( m_k^2 - 1 \) gaps of length at least \( m_k^{-2} \). Requires at least \( \frac{1}{2} L^{-1} \) of length in the domain of \( f \) by Lipschitz condition.
So for \( \Gamma \) to contain \( E \) there would have to be infinite length in the domain of \( f \), which is a contradiction.
Not contained in a rectifiable curve: a countable compact set with one accumulation point

For each $k \geq 2$, choose $m_k = k^2$ so that $\sum_{k=2}^{\infty} m_k^{-1} < \infty$. Arrange squares $S_k$ with side length $m_k^{-1}$ so that one side of each square lies on a given line; separate $S_k$ and $S_{k+1}$ by distance $m_k^{-1}$. Let $V_k$ be collection of $m_k^2$ points in $S_k$ separated by distance at least $m_k^{-2}$. Let $E$ be the closure of $\bigcup_{k=2}^{\infty} V_k$.

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To contain $V_k$, the curve $\Gamma$ must cross $m_k^2 - 1$ gaps of length at least $m_k^{-2}$. Requires at least $\frac{1}{2}L^{-1}$ of length in the domain of $f$ by Lipschitz condition.

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Not contained in a rectifiable curve: a countable compact set with one accumulation point

For each $k \geq 2$, choose $m_k = k^2$ so that $\sum_{k=2}^{\infty} m_k^{-1} < \infty$. Arrange squares $S_k$ with side length $m_k^{-1}$ so that one side of each square lies on a given line; separate $S_k$ and $S_{k+1}$ by distance $m_k^{-1}$. Let $V_k$ be collection of $m_k^2$ points in $S_k$ separated by distance at least $m_k^{-2}$. Let $E$ be the closure of $\bigcup_{k=2}^{\infty} V_k$.

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a countable compact set with one accumulation point

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Suppose $\Gamma = f([0, 1]) \supset E$ for some $f$ with $|x - y| \geq L^{-1}|f(x) - f(y)|$

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So for $\Gamma$ to contain $E$ there would have to be infinite length in the domain of $f$, which is a contradiction.
Not contained in a rectifiable curve:
a countable compact set with one accumulation point

For each \( k \geq 2 \), choose \( m_k = k^2 \) so that \( \sum_{k=2}^{\infty} \frac{1}{m_k} < \infty \). Arrange squares \( S_k \) with side length \( \frac{1}{m_k} \) so that one side of each square lies on a given line; separate \( S_k \) and \( S_{k+1} \) by distance \( \frac{1}{m_k} \). Let \( V_k \) be collection of \( m_k \) points in \( S_k \) separated by distance at least \( \frac{1}{m_k^2} \). Let \( E \) be the closure of \( \bigcup_{k=2}^{\infty} V_k \).

Suppose \( \Gamma = f([0,1]) \supset E \) for some \( f \) with \( |x - y| \geq L^{-1} |f(x) - f(y)| \).

To contain \( V_k \), the curve \( \Gamma \) must cross \( m_k^2 - 1 \) gaps of length at least \( \frac{1}{m_k^2} \). Requires at least \( \frac{1}{2} L^{-1} \) of length in the domain of \( f \) by Lipschitz condition.

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To contain \( V_k \), the curve \( \Gamma \) must cross \( m_k^2 - 1 \) gaps of length at least \( m_k^{-2} \). Requires at least \( \frac{1}{2}L^{-1} \) of length in the domain of \( f \) by Lipschitz condition.

So for \( \Gamma \) to contain \( E \) there would have to be infinite length in the domain of \( f \), which is a contradiction.
For any nonempty set $E \subset \mathbb{R}^n$ and bounded “window” $Q \subset \mathbb{R}^n$, the **Jones beta number** of $E$ in $Q$ is

$$\beta_E(Q) := \inf_{\text{line } \ell} \sup_{x \in E \cap Q} \frac{\text{dist}(x, \ell)}{\text{diam } Q} \in [0, 1].$$

If $E \cap Q = \emptyset$, we also define $\beta_E(Q) = 0$. 
Theorem (P. Jones (1990), K. Okikiolu (1992))

Let $E \subset \mathbb{R}^n$ be a bounded set. Then $E$ is contained in a rectifiable curve if and only if

$$S_E := \sum_{\text{dyadic } Q} \beta_E(3Q)^2 \text{diam } Q < \infty$$

More precisely:

1. If $S_E < \infty$, then there is a curve $\Gamma \supset E$ such that
   $$\mathcal{H}^1(\Gamma) \lesssim_n \text{diam } E + S_E.$$
2. If $\Gamma$ is a curve containing $E$, then $\text{diam } E + S_E \lesssim_n \mathcal{H}^1(\Gamma)$. 
Open Problem #5

Theorem (P. Jones (1990), K. Okikiolu (1992))

Let $E \subset \mathbb{R}^n$ be a bounded set. Then $E$ is contained in a rectifiable curve if and only if

$$S_E := \sum_{\text{dyadic } Q} \beta_E(3Q)^2 \text{diam } Q < \infty$$

More precisely:

1. If $S_E < \infty$, then there is a curve $\Gamma \supset E$ such that $\mathcal{H}^1(\Gamma) \lesssim_n \text{diam } E + S_E$.

2. If $\Gamma$ is a curve containing $E$, then $\text{diam } E + S_E \lesssim_n \mathcal{H}^1(\Gamma)$.

Find characterizations of subsets of other nice families of sets.
Hölder Traveling Salesman Theorem

Theorem (B, Naples, Vellis 2018)

For all $s > 1$, there exists a constant $\beta_0 = \beta_0(s, n) > 0$ such that:

If $E \subset \mathbb{R}^n$ is a bounded set and

$$\sum_{\substack{Q \text{ dyadic} \\ \beta_E(3Q) \geq \beta_0}} (\text{diam } Q)^s < \infty,$$

then $E$ is contained in a $(1/s)$-Hölder curve.

Corollary

Assume $s > 1$. If $E \subset \mathbb{R}^n$ is a bounded set and

$$\sum_{\substack{Q \text{ dyadic} \\ \text{side } Q \leq 1}} \beta_E(3Q)^2(\text{diam } Q)^s < \infty,$$

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Remarks

- There is a version of the theorem in infinite-dimensional Hilbert space
- Construction of approximating curves $\Gamma_k$ are similar to case $s = 1$
- But unlike the case $s = 1$, we do not have Ważewski’s theorem!!!
- So we have reimagine Jones’ proof of the traveling salesman construction and build explicit parameterization of the $\Gamma_k$
- The condition is not necessary (e.g. fails for a Sierpinski carpet)
Part I. Curves

Part II. Subsets of Curves

Part III. Rectifiability of Measures
Measure Theorist’s Traveling Salesman Problem

Given a finite Borel measure $\mu$ on $\mathbb{R}^n$ with bounded support ($\iff \mu(\mathbb{R}^n \setminus B) = 0$ for some bounded set $B$), decide whether or not $\mu$ is carried by a rectifiable curve.

If so, construct a rectifiable curve $\Gamma$ carrying $\mu$, i.e. $\mu(\mathbb{R}^n \setminus \Gamma) = 0$.

This is solved for

- $\mu$ such that $\mu(B(x, r)) \sim r$ for $x \in \text{spt} \mu$ by Lerman (2003)
- $\mu$ any finite Borel measure by B and Schul (2017)
Non-homogeneous $L^2$ Jones $\beta$ numbers

Let $\mu$ be a Radon measure on $\mathbb{R}^n$. For every cube $Q$, define

$$\beta_2(\mu, 3Q) = \inf_{L} \beta_2(\mu, 3Q, L) \in [0, 1],$$

where

$$\beta_2(\mu, 3Q, L)^2 = \int_{3Q} \left( \frac{\text{dist}(x, L)}{\text{diam } 3Q} \right)^2 \frac{d\mu(x)}{\mu(3Q)}.$$

“Non-homogeneous” refers to the normalization $1/\mu(3Q)$.
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\[\beta_2 = 0\]

\[\beta_2 \text{ small}\]

\[\beta_2 \sim 1\]
Traveling Salesman for Ahlfors Regular Measures

Theorem (Lerman 2003)

Let $\mu$ be a finite measure on $\mathbb{R}^n$ with bounded support. Assume that

$$\mu(B(x, r)) \sim r \quad \text{for all } x \in \text{spt } \mu \text{ and } 0 < r \leq 1.$$  

Then there is a rectifiable curve $\Gamma$ such that $\mu(\mathbb{R}^n \setminus \Gamma) = 0$ if and only if

$$\sum_{\text{dyadic } Q} \beta_2(\mu, 3Q)^2 \text{diam } Q < \infty.$$ 

Theorem (Martikainen and Orponen 2018)

There exists a Borel probability $\nu$ on $\mathbb{R}^2$ with bounded support such that

$$\sum_{\text{dyadic } Q} \beta_2(\nu, 3Q)^2 \text{diam } Q < \infty$$

but $\nu$ is purely 1-unrectifiable, i.e. $\nu(\Gamma) = 0$ for every rectifiable curve $\Gamma$. 
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Anisotropic $L^2$ Jones $\beta$ numbers (B-Schul 2017)

Given dyadic cube $Q$ in $\mathbb{R}^n$, $\Delta^*(Q)$ denotes a subdivision of $Q^* = 1600\sqrt{n}Q$ into dyadic cubes $R$ of same / previous generation as $Q$ s.t. $3R \subseteq Q^*$.

For every Radon measure $\mu$ on $\mathbb{R}^n$ and every dyadic cube $Q$, we define $\beta^\ast\ast_2(\mu, Q)^2 = \inf_{\text{line } L} \max_{R \in \Delta^*(Q)} \beta_2(\mu, 3R, L)^2$, where

$$\beta_2(\mu, 3R, L)^2 = \int_{3R} \left( \frac{\text{dist}(x, L)}{\text{diam } 3R} \right)^2 \frac{d\mu(x)}{\mu(3R)}$$
Traveling Salesman Theorem for Measures

Theorem (B and Schul 2017)

Let $\mu$ be a finite measure on $\mathbb{R}^n$ with bounded support. Then there is a rectifiable curve $\Gamma$ such that $\mu(\mathbb{R}^n \setminus \Gamma) = 0$ if and only if

$$\sum_{\text{dyadic } Q} \beta^{**}_2(\mu, Q)^2 \text{diam } Q < \infty.$$  

- Proof uses both halves of the traveling salesman theorem curves
- For the sufficient half, need extension of the traveling salesman construction without requirement $V_{k+1} \supset V_k$ (see B-Schul 2017)
- Using similar techniques, we can also get a characterization of countably 1-rectifiable Radon measures
Identification of 1-rectifiable Radon measures

For any Radon measure $\mu$ on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, the lower density is:

$$D^1(\mu, x) \equiv \liminf_{r \downarrow 0} \frac{\mu(B(x, r))}{2r} \in [0, \infty]$$

and the anisotropic square function is:

$$J^*_2(\mu, x) \equiv \sum_{\text{dyadic } Q \text{ such that } \mu(Q) \leq \beta_2^* \text{ and } \text{diam } Q \leq 1} \beta_2^* \mu(Q)^2 \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x) \in [0, \infty]$$

Theorem (B and Schul 2017)

If $\mu$ is a Radon measure on $\mathbb{R}^n$, then

$\mu \subseteq \{x : D^1(\mu, x) > 0 \text{ and } J^*_2(\mu, x) < \infty\}$ is countably 1-rectifiable

$\mu \subseteq \{x : D^1(\mu, x) = 0 \text{ or } J^*_2(\mu, x) = \infty\}$ is purely 1-unrectifiable
Open Problem #6

Given a measurable space \((X, \mathcal{M})\) and a family of sets \(\mathcal{N}\), every \(\sigma\)-finite measure \(\mu\) on \(\mathbb{R}^n\) decomposes as \(\mu = \mu_\mathcal{N} + \mu_\perp\mathcal{N}\), where

- \(\mu_\mathcal{N}\) is carried by \(\mathcal{N}\): \(\mu_\mathcal{N}(X \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0\) for some \(\Gamma_i \in \mathcal{N}\)
- \(\mu_\perp\mathcal{N}\) is singular to \(\mathcal{N}\): \(\mu_\perp\mathcal{N}(\Gamma) = 0\) for all \(\Gamma \in \mathcal{N}\).

Identification Problem:
Given \((X, \mathcal{M})\), \(\mathcal{N} \subset \mathcal{M}\), and of \(\mathcal{F}\) a family of \(\sigma\)-finite measures on \(\mathcal{M}\), find properties \(P(\mu, x)\) and \(Q(\mu, x)\) defined for all \(\mu \in \mathcal{F}\) and \(x \in X\) such that
\[
\mu_\mathcal{N} = \mu \perp \{x : P(\mu, x)\} \quad \text{and} \quad \mu_\perp\mathcal{N} = \mu \perp \{x : Q(\mu, x)\}
\]

An important case is \(X = \mathbb{R}^n\), \(\mathcal{N}\) is Lipschitz images of \(\mathbb{R}^m\) \((m \geq 2)\), and \(\mathcal{F}\) is Radon measures on \(\mathbb{R}^n\)
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An important case is \(X = \mathbb{R}^n\), \(\mathcal{N}\) is Lipschitz images of \(\mathbb{R}^m\) (\(m \geq 2\)), and \(\mathcal{F}\) is Radon measures on \(\mathbb{R}^n\).
Criteria for fractional rectifiability

A model for **fractional rectifiability** based on Hölder continuous images of $\mathbb{R}^m$ in $\mathbb{R}^n$ was proposed by Martín and Mattila (1993,2000).

**Theorem (B, Vellis 2018)**

Let $s > 1$ and $m \leq t < s$. Assume that $\mu$ is a Radon measure on $\mathbb{R}^n$ such that

$$0 < \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r^t} \leq \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^t} < \infty \quad \mu\text{-a.e. } x.$$

Then $\mu$ is carried by $(m/s)$-Hölder continuous images of $[0, 1]^m$.

**Theorem (B, Naples, Vellis 2018)**

Let $s > 1$. Assume that $\mu$ is a Radon measure on $\mathbb{R}^n$ such that

$$\limsup_{r \downarrow 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty \quad \mu\text{-a.e. } x, \quad \text{and}$$

$$\int_0^1 \beta_2(\mu, B(x,r))^\alpha \frac{r^s}{\mu(B(x,r))} \frac{dr}{r} < \infty \quad \mu\text{-a.e. } x.$$

Then $\mu$ is carried by $(1/s)$-Hölder curves.
Thank you for listening!