This is the last homework assignment.

Due In Class: Thursday, December 4.

Reading: Read Chapter 7.

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A: Let $[a_k, b_k]$ be a sequence of closed intervals shrinking to x_0, i.e.

$$\bigcap_{k=1}^{\infty} [a_k, b_k] = \{x_0\}.$$

Prove that if $f(x)$ is Riemann integrable near x_0 and continuous at x_0, then

$$f(x_0) = \lim_{k\to\infty} \frac{1}{b_k - a_k} \int_{a_k}^{b_k} f(x)\, dx.$$

Hint: Use ideas from the proof of Theorem 6.20.

Problem B: Exercise 7.2 (multiplication only) and 7.3

Problem C: Exercise 7.9

Problem D: Suppose $\alpha \in (0,1]$ and $f_k : [0,2] \to \mathbb{R}$ for all $k \geq 1$. Prove that if $f_k(1) = 1$ for all $k \geq 1$ and there exists a constant $C \in [0,\infty)$ such that

$$|f_k(x) - f_k(y)| \leq C|x-y|^{\alpha} \quad \text{for all } x, y \in [0,2] \text{ and } k \geq 1, \tag{$\star$}$$

then $(f_k)_{k=1}^{\infty}$ has a uniformly convergent subsequence whose limit f also satisfies $(\star)$.

Remark: A function satisfying $(\star)$ is called Hölder continuous of order α.

Problem E: Exercise 7.18

The following statement is a corrected version of Portfolio Problem 3 that should be included in the final draft of your writing portfolio.

Portfolio Problem 3: Let X be a metric space. A *(closed) curve* in X is a continuous map $\gamma : [a,b] \to X$. The *length* $\ell(\gamma)$ of γ is given by

$$\ell(\gamma) := \sup\left\{ \sum_{i=1}^{n} d_X(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 \leq t_1 \leq \cdots \leq t_n = b, n \geq 1 \right\} \in [0,\infty].$$

If $\ell(\gamma) < \infty$, then γ is called a *rectifiable curve*. For every increasing continuous map from $[c,d]$ *onto* $[a,b]$, the curve $\gamma \circ r : [c,d] \to X$ is called a *reparameterization* of γ.

(a) Prove that if γ is Lipschitz continuous, then γ is a rectifiable curve.

(b) Prove that if γ is a rectifiable curve, then there exists a rectifiable curve $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ and an increasing continuous map s from $[a,b]$ onto $[0,\ell(\gamma)]$ such that $\gamma = \tilde{\gamma} \circ s$ and

$$\ell(\tilde{\gamma}_{[p,q]}) = q - p \quad \text{for all } 0 \leq p \leq q \leq \ell(\gamma).$$

The curve $\tilde{\gamma}$ is called the *arc-length parameterization* of γ.