Math 4110/5110, Introduction to Modern Analysis, Fall 2014 — Homework #10

Due In Class: Thursday, November 13.

Reading: Start reading Chapter 6.

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A: Exercise 5.12

Problem B: Exercise 5.13 *You may use without proof the fact that the sine and cosine functions* $\sin x$ *and* $\cos x$ *are differentiable on* $\mathbb{R}$, $D_x(\sin x) = \cos x$, *and* $D_x(\cos x) = -\sin x$.

Problem C: Exercise 5.15

Problem D: Exercise 6.2

Problem E: Exercise 6.4

The following problems should be included in the **writing portfolio**, a **draft** of which is **due in class on Thursday, November 13th**. The portfolio should be typeset (e.g. using LaTeX, etc.) and should follow the conventions outlined in 'Some Remarks on Writing Mathematical Proofs' by John M. Lee (available by link on the course webpage). You may discuss the solutions of portfolio problems with your classmates, but all write-ups should be completed independently. Feedback on your portfolio is available at any time during the instructor's office hours.

Portfolio Problem 1: Give a complete, self-contained proof of Theorem 4.9 in the textbook.

Portfolio Problem 2: An extended real-valued function $\eta : [0, +\infty) \to [0, +\infty]$ is called a *modulus of continuity* if $\eta(0) = 0$ and $\lim_{t\to 0} \eta(t) = 0$. A function $f : X \to Y$ between metric spaces *admits a modulus of continuity inequality* if there is a modulus of continuity η such that

$$d_Y(f(x), f(x')) \leq \eta\left(d_X(x, x')\right) \quad \text{for all } x, x' \in X.$$

Prove that $f : X \to Y$ is uniformly continuous if and only if f admits a modulus of continuity inequality.

Portfolio Problem 3: Let X be a metric space. A *(closed) curve* in X is a continuous map $\gamma : [a, b] \to X$. The *length* $\ell(\gamma)$ of γ is given by

$$\ell(\gamma) := \sup \left\{ \sum_{i=1}^{n} d_X(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 \leq t_1 \leq \cdots \leq t_n = b, n \geq 1 \right\} \in [0, \infty].$$

If $\ell(\gamma) < \infty$, then γ is called a *rectifiable curve.* For every increasing homeomorphism $r : [c, d] \to [a, b]$, the curve $\gamma \circ r : [c, d] \to X$ is called a *reparameterization* of γ.

(a) Prove that if γ is Lipschitz continuous, then γ is a rectifiable curve.

(b) Prove that if γ is a rectifiable curve, then there exists an increasing homeomorphism $s : [0, \ell(\gamma)] \to [a, b]$ such that $\gamma \circ s$ is Lipschitz and $\ell((\gamma \circ s)|_{[p,q]}) = q - p$ for all $0 \leq p \leq q \leq \ell(\gamma)$. The curve $\gamma \circ s$ is called the *arc-length parameterization* of γ.

Portfolio Problem 4: *This is the last portfolio problem.* Exercises 5.26 and 5.27 in the textbook.