MATH 3150 PORTFOLIO PROBLEMS

PROF. MATTHEW BADGER

Instructions: Prepare a solution for each portfolio problem below by the indicated due date. Each problem should be typed or handwritten neatly on its own sheet(s) of paper. Your solutions should follow the conventions described in John M. Lee's *Some Remarks on Writing Mathematical Proofs*, a link to which is posted on the class website. (Remember to start your solutions with clear theorem statements.) Keep returned drafts and prepare revisions to be submitted together on the last day of class.

1. Problems Due Friday, September 29

Problem 1 (corrected): Prove that for all $x, z \in \mathbb{Q}$ with $0 < x < z$, there exists $y \in \mathbb{Q}$ such that $x < y < z$ and y has a rational square root (i.e. $y = w^2$ for some $w \in \mathbb{Q}$.)

Problem 2: *Choose (A) or (B)*

(A) Let $(X, +, \cdot, <)$ be an ordered field. Recall from class that a *cut* (L, R) of X is a pair of nonempty sets $L, R \subseteq X$ such that $X = L \cup R$, $L \cap R = \emptyset$, and $x < y$ for all $x \in L$ and $y \in R$. We say that X has the *number cutting property* if for every cut (L, R) of X, there exists $x_0 \in X$ such that

$$L = \{x \in X : x \leq x_0\} \quad \text{and} \quad R = \{y \in X : y > x_0\}$$

or

$$L = \{x \in X : x < x_0\} \quad \text{and} \quad R = \{y \in X : y \geq x_0\}.$$

Prove that if X has the number cutting property, then X is a complete field.

(B) Let $(X, +, \cdot, <)$ be an ordered field. Prove the *infimum approximation property*: if $A \subseteq X$ is bounded below and m is a lower bound for A, then $m = \inf A$ if and only if for all $\varepsilon \in X$ with $\varepsilon > 0$, there exists $a \in A$ such that $m \leq a < m + \varepsilon$.

Problem 3: Given nonempty sets $A, B \subseteq \mathbb{R}$, define $A + B := \{a + b : a \in A, b \in B\}$. Prove that if A and B are bounded below, then $A + B$ is bounded below, and moreover, $\inf(A + B) = \inf A + \inf B$.

Date: September 29, 2017.

2. Problems Due Friday, October 20

Problem 4: Prove that every open set $O \subseteq \mathbb{R}$ can be written as a union of finitely or countably many open intervals (a_i, b_i).

Problem 5: A sequence (a_n) is called *subadditive* if $a_{m+n} \leq a_m + a_n$ for all $m, n \in \mathbb{N}$. Prove that if (a_n) is a subadditive sequence of positive real numbers, then $\left(\dfrac{a_n}{n}\right)$ converges. [Hint: See Exercises 2.3.8, 2.3.9]

Problem 6: Let (a_n) be a sequence of real numbers. We say that $c \in \mathbb{R}$ is a *cluster point* of (a_n) if there exists a subsequence (a_{n_k}) of (a_n) such that

$$\lim_{k \to \infty} a_{n_k} = c.$$

For each sequence, define $\mathscr{C}(a_n) := \{c \in \mathbb{R} : c \text{ is a cluster point of } (a_n)\}$, the set of all cluster points of (a_n). Prove that for all $m \in \mathbb{N}$, there exists a sequence (a_n) such that $\#\mathscr{C}(a_n) = m$.