Hölder parameterizations of Bedford-McMullen carpets and connected IFS

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Joint work with Vyron Vellis (Tennessee)

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A curve $\Gamma$ in a metric space $X$ is a **continuous image** of $[0, 1]$: There exists a continuous map $f : [0, 1] \to X$ such that $\Gamma = f([0, 1])$

A continuous map $f$ with $\Gamma = f([0, 1])$ is called a **parameterization** of $\Gamma$

- There are curves which do not have a 1-1 parameterization
- There are curves which have topological dimension $> 1$
- The modulus of continuity of a parameterization is a proxy for the size/regularity/complexity of a curve
Two characterizations (early 20th century)

**Hahn-Mazurkiewicz Theorem** A set $\Gamma$ in a metric space is a **curve** if and only if $\Gamma$ is compact, connected, locally connected.

**Ważewski Theorem** A set $\Gamma$ in a metric space is a **rectifiable curve** iff $\Gamma$ is a **Lipschitz curve** iff $\Gamma$ is compact, connected, and $\mathcal{H}^1(\Gamma) < \infty$

- In $\mathbb{R}^n$: Lipschitz curves (also called *rectifiable curves*) admit unique tangent lines at $\mathcal{H}^1$-a.e. point by Rademacher’s theorem.
- Note compact, connected, and $\mathcal{H}^1(\Gamma) < \infty$ implies $\Gamma$ locally connected! This can fail for sets with $\sigma$-finite length (e.g. a topologist’s comb).

$f$ is Lipschitz if $\exists L < \infty$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y$

$\mathcal{H}^s$ denotes the $s$-dimensional Hausdorff measure
Two characterizations (early 20th century)

**Hahn-Mazurkiewicz Theorem** A set \( \Gamma \) in a metric space is a curve if and only if \( \Gamma \) is compact, connected, locally connected.

![Image of a curve](image1)

**Ważewski Theorem** A set \( \Gamma \) in a metric space is a rectifiable curve iff \( \Gamma \) is a Lipschitz curve iff \( \Gamma \) is compact, connected, and \( \mathcal{H}^1(\Gamma) < \infty \)

- In \( \mathbb{R}^n \): Lipschitz curves (also called rectifiable curves) admit unique tangent lines at \( \mathcal{H}^1 \)-a.e. point by Rademacher’s theorem
- Note compact, connected, and \( \mathcal{H}^1(\Gamma) < \infty \) implies \( \Gamma \) locally connected! This can fail for sets with \( \sigma \)-finite length (e.g. a topologist’s comb)

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\( \mathcal{H}^s \) denotes the s-dimensional Hausdorff measure
What about higher-dimensional curves?
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Open Problem (#1)

For each real \( s \in (1, \infty) \), characterize curves \( \Gamma \subset \mathbb{R}^n \) with \( \mathcal{H}^s(\Gamma) < \infty \)

Open Problem (#2)

For each real \( s \in (1, \infty) \), characterize \((1/s)\)-Hölder curves, i.e. sets that can be presented as \( h([0, 1]) \) for some map \( h : [0, 1] \to \mathbb{R}^n \) with

\[
|h(x) - h(y)| \leq H|x - y|^{1/s}
\]

- Every \((1/s)\)-Hölder curve has \( \mathcal{H}^s(\Gamma) < \infty \) (exercise)
- Example (B-Naples-Vellis, Adv. Math. 2019): For every \( s \in (1, n) \), \( \exists \) a curve \( \Gamma \subset \mathbb{R}^n \) that is \( s \)-Ahlfors regular, \( \mathcal{H}^s(\Gamma \cap B(x, r)) \approx r^s \), but \( \Gamma \) is not a \((1/s)\)-Hölder curve.
Why?

- There are many dimensions between 1 and 2
- Lipschitz surfaces are Hölder curves:
  to study the former, first study the latter
- Martín and Mattila (1993,2000) developed a portion of
  Besicovitch’s fine theory of 1-sets in $\mathbb{R}^2$ works for $s$-sets in $\mathbb{R}^n$
  using Hölder curves as a replacement for rectifiable curves
- There exist metric spaces without rectifiable curves that are
  Hölder path connected
- Modulus of path families makes sense for Hölder curves
- Random settings: Brownian motion, rough paths theory
- Possible tool for singular integrals in codimension $> 1$
Sufficient conditions for Hölder curves

Theorem (Remes 1998)

Let $S \subset \mathbb{R}^n$ be a self-similar set satisfying the open set condition. If $S$ is connected, then $S$ is a $(1/s)$-Hölder curve, $s = \dim_{H} S$.

A set $E \subset \mathbb{R}^n$ is $\beta$-flat if for every $x \in E$ and $r > 0$, there exists a line $\ell$ such that $\text{dist}(x, \ell) \leq \beta r$ for all $x \in E \cap B(x, r)$.


There exists a universal constant $\beta_0 \in (0, 1)$ such that if $E \subset \mathbb{R}^n$ is $\beta_0$-flat, connected, compact, $\mathcal{H}^s(E) < \infty$, and $\mathcal{H}^s(E \cap B(x, r)) \gtrsim r^s$, then $E$ is a $(1/s)$-Hölder curve.
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Iterated Function Systems (Quick Review)

Let $X$ be a complete, separable metric space. A **contraction** in $X$ is a map $\phi : X \to X$ with Lipschitz constant $\text{Lip}(\phi)$ strictly less than 1

$$\text{Lip}(\phi) = \inf\{L \geq 0 : \text{dist}(\phi(x), \phi(y)) \leq L \text{dist}(x, y)\}$$

**Hutchinson’s Theorem** For every finite family $\mathcal{F}$ of contractions in $X$, there exists a unique compact set $K \subset X$ such that $K = \bigcup_{\phi \in \mathcal{F}} \phi(K)$.

- $\mathcal{F}$ is called an **iterated function system**
- $K = K_{\mathcal{F}}$ is called the **attractor** of $\mathcal{F}$
- $\mathcal{H}^s(K) < \infty$ where $s$ is the **similarity dimension** of $\mathcal{F}$, i.e.

$$s \geq 0 \text{ is the unique number such that } \sum_{\phi \in \mathcal{F}} \text{Lip}(\phi)^s = 1$$

- If each $\phi \in \mathcal{F}$ is a similarity, i.e. $\text{dist}(\phi(x), \phi(y)) = \lambda_\phi \text{dist}(x, y)$, then we call $K$ a **self-similar set**
"Iterated Function Systems", Google Image Search, 3pm (Atlanta) on 9/13/2019
IFS with Connected Attractors

Theorem (Hata 1985)

Let $\mathcal{F}$ be an IFS over a complete metric space. If $K_{\mathcal{F}}$ is connected, then $K_{\mathcal{F}}$ is path connected and locally connected. Thus, $K_{\mathcal{F}}$ is a curve.

Let $\mathcal{F}$ be an IFS over a complete metric space; let $s$ be the similarity dimension of $\mathcal{F}$.

Theorem (B-Vellis, arXiv October 2019 (I’m optimistic))

If $K_{\mathcal{F}}$ is connected, then $K_{\mathcal{F}}$ is $(1/s)$-Hölder path connected.

Theorem (B-Vellis, arXiv October 2019)

If $K_{\mathcal{F}}$ is connected, then $K_{\mathcal{F}}$ is a $(1/\alpha)$-Hölder curve for every $\alpha > s$.

- Second theorem is a corollary of the first, viewing $K_{\mathcal{F}}$ as leaves of a tree with $(1/s)$-Hölder edges (cf. B-Vellis JGA 2019)
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Remes’ parameterization of self-similar sets

Let $\mathcal{F}$ be an IFS over a complete metric space $X$ that is \textbf{generated by similarities}; let $s$ be the similarity dimension dimension of $\mathcal{F}$.

\textbf{Theorem}

\textit{If $K_\mathcal{F}$ is connected and $\mathcal{H}^s(K_\mathcal{F}) > 0$, then $K_\mathcal{F}$ is a $(1/s)$-Hölder curve.}

- Remes (1998) proved this when $X = \mathbb{R}^n$, where $\mathcal{H}^s(K_\mathcal{F}) > 0 \iff \text{SOSC} \iff \text{OSC} \implies \dim_H K_\mathcal{F} = s$ (Schief 1994)
- In complete metric spaces: $\mathcal{H}^s(K_\mathcal{F}) > 0 \implies \text{SOSC} \implies \dim_H K_\mathcal{F} = s$ (Schief 1996)
- Self-similar implies $K_\mathcal{F}$ with $\mathcal{H}^s(K_\mathcal{F}) > 0$ are $s$-Ahlfors regular
- To prove thm, embed $K_\mathcal{F}$ in $\ell_\infty$ and repeat Remes’ original proof

Open Set Condition: $\exists U$ open s.t. $\phi(U) \subseteq U$, $\phi(U) \cap \psi(U) = \emptyset$ for distinct $\phi, \psi \in \mathcal{F}$. Strong Open Set Condition: also $U \cap K_\mathcal{F} \neq \emptyset$. 
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Case Study: Bedford-McMullen Carpets

Let $\Sigma$ be a Bedford-McMullen carpet (see diagram).

- **Similarity dimension is**
  \[
  s = \log_n (t_1 + \cdots + t_n)
  \]

McMullen (1984)

- $\dim_H \Sigma = \log_n \left( \sum_{j=1}^n t_j \right) \log_m(n)$
- $\dim_M \Sigma = \log_n(r) + \log_m \left( \sum_{j=1}^n t_j / r \right)$

Mackay (2011)

- If $m < n$ (self-affine), then
  \[
  \dim_A \Sigma = \log_n(r) + \log_m(t)
  \]
Case Study: Bedford-McMullen Carpets
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Theorem (B-Vellis arXiv October 2019)

Let $\Sigma \subset [0, 1]^2$ be a connected Bedford-McMullen carpet.

- If $\Sigma$ is a line, $\Sigma$ is (trivially) a $1$-Hölder curve
- If $\Sigma$ is the square, $\Sigma$ is (well-known to be) a $(1/2)$-Hölder curve
- Otherwise, $\Sigma$ is a $(1/s)$-Hölder curve, $s$ similarity dimension

The exponents are sharp (they cannot be increased).

- Idea: Lift $\Sigma$ to a self-similar set $K$ in $([0, 1]^2, d)$ equipped with a partially snowflaked metric $d$ via a Lipschitz map $F : K \rightarrow \Sigma$. Use Remes’ theorem upstairs to parameterize $K$. Then descend.
- If $X$ doubling: $\mathcal{H}^s(K) > 0 \iff SOSC$ (Stella 1992 / Schief 1996)
- When does an IFS admit a Lipschitz lift to self-similar set in doubling space (or a $\beta$-space)?
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Self-similar vs self-affine carpets and a conjecture

Σ be a connected self-similar / self-affine Bedford-McMullen carpet, 
\( D = \text{Hausdorff dimension}, s = \text{similarity dimension} \) (sometimes \( s > 2 \)!

Self-Similar

\[
D = s, \ s \in [1, 2] \\
0 < \mathcal{H}^s(\Sigma) < \infty 
\]

Self-Affine

\[
D < s, \ s \in [1, \infty) \\
\mathcal{H}^s(\Sigma) = 0 
\]

Conjecture (B 2018): If \( \Gamma \subset \mathbb{R}^n \) is a \((1/s)\)-Hölder curve with \( \mathcal{H}^s(\Gamma) > 0 \), then at \( \mathcal{H}^s\)-a.e. \( x \in \Gamma \), all geometric blow-ups (tangent sets) of \( \Gamma \) at \( x \) are “self-similar” \((1/s)\)-Hölder images of \( \mathbb{R} \).
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<table>
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Conjecture (B 2018): If $\Gamma \subset \mathbb{R}^n$ is a $(1/s)$-Hölder curve with $\mathcal{H}^s(\Gamma) > 0$, then at $\mathcal{H}^s$-a.e. $x \in \Gamma$, all geometric blow-ups (tangent sets) of $\Gamma$ at $x$ are “self-similar” $(1/s)$-Hölder images of $\mathbb{R}$
Related and Future Work

A related, but different problem: What sets in a metric space are contained in a \((1/s)\)-Hölder curve?

- New result in quasiconvex metric spaces: Balogh and Züst arXiv 2019 (in summer)

Future work:

- We need to find good necessary conditions for a set to be (contained in) a \((1/s)\)-Hölder curve
- Applications to Geometry of Measures (cf. Lisa Naples’ talk), Metric Geometry. Random geometry? Singular integrals?
Thank you for listening!