DIVISION ALGEBRAS OVER THE REAL NUMBERS

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Abstract. Midterm project for Math 2501, Abstract Algebra II, at the University of Pittsburgh. After constructing the quaternions and octonions, we establish the classifications of the associative, normed and alternative division algebras over the real numbers.

1. Preliminaries

In this article, we introduce division algebras over the real numbers. Our aim is to establish theorems of Frobenius, Hurwitz and Zorn, which classify the associative, normed and alternative real division algebras, up to isomorphism. For now, we postpone historical remarks to §5, preferring to delve straight into the mathematics. Let us first agree on some definitions.

Definition 1.1. Let $\mathbf{F}$ be a field. An *algebra* A over $\mathbf{F}$ is a pair (A, m), where A is a finite-dimensional vector space over $\mathbf{F}$ and *multiplication* $m : A \times A \to A$ is an $\mathbf{F}$-bilinear map; that is, for all $\lambda \in \mathbf{F}$, $x, y, z \in A$,

$$m(x, \lambda y + z) = \lambda m(x, y) + m(x, z),$$
$$m(\lambda x + y, z) = \lambda m(x, z) + m(y, z).$$

Two algebras (A, m) and (B, n) over $\mathbf{F}$ are said to be *isomorphic* if there is an invertible map $\phi : A \to B$ such that for all $x, y \in A$,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

When clear from the context, we write $m(x, y) = xy$ for all $x, y \in A$. ⊣

Definition 1.2. Let A be an algebra over $\mathbf{F}$. Then A is said to be

(1) *alternative* if $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in A$,
(2) *associative* if $x(yz) = (xy)z$ for all $x, y, z \in A$,
(3) *commutative* if $xy = yx$ for all $x, y \in A$, and
(4) *unital* if there is a $1 \in A$ such that $x1 = x = 1x$ for all $x \in A$.

If A is unital, then the *identity* 1 is uniquely determined. ⊣

Remark 1.3. Warning! Unless stated explicitly we do not assume that an algebra A is alternative, associative, commutative or unital. ⊣

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Lemma 1.4. *Let A be an alternative algebra over $\mathbf{F}$. Then A obeys*

(1) *the* flexible *law: $x(yx) = (xy)x$ for all $x, y \in A$, and*
(2) *the* Moufang identity*: $(zx)(yz) = z(xy)z$ for all $x, y, z \in A$.*

If we define x^n for $n \in \mathbb{Z}^+$ recursively by $x^1 = x$ and $x^{n+1} = x^n x$, then

(3) *A is* power-associative*: $x^m x^n = x^{m+n}$ for all $x \in A$, $m, n \in \mathbb{Z}^+$.*

Proof. The associator $[x, y, z] \mapsto x(yz) - (xy)z$ for all $x, y, z \in A$ is a trilinear map $A^3 \to A$. Since A is alternative, the associator *alternates*:

$$[x, y, z] = -[y, x, z] = -[x, z, y] = -[z, y, x] \quad \text{for all } x, y, z \in A.$$

To prove the first equality, for instance, observe that

$$\begin{aligned}
0_A &= [x + y, x + y, z] && (A \text{ alternative})\\
&= [x, x, z] + [x, y, z] + [y, x, z] + [y, y, z] && (\text{linearity of } [\cdot, \cdot, \cdot])\\
&= [x, y, z] + [y, x, z]. && (A \text{ alternative})
\end{aligned}$$

Hence, $[x, y, z] = -[y, x, z]$ for all $x, y, z \in A$. The remaining equalities follow similarly.

For (1), observe that $[x, y, x] = -[y, x, x] = 0_A$ (since A alternative). Thus, $x(yx) - (xy)x = 0$, or equivalently, $x(yx) = (xy)x$ for all $x, y \in A$.

For (2), observe first that, when A is alternative, repeated use of the identities above yields:

$$\begin{aligned}
(zx)(yz) - ((zx)y)z = [zx, y, z] = [y, z, zx] &= y(z^2x) - (yz)(zx)\\
&= y(z^2x) - [yz, z, x] - (yz^2)x\\
&= [y, z^2, x] - [yz, z, x] = [y, z^2, x] - [x, yz, z]\\
&= [y, z^2, x] - x(yz^2) + (x(yz))z\\
&= [y, z^2, x] + [x, y, z]z - [x, y, z^2] = [x, y, z]z.
\end{aligned}$$

Therefore, if A is alternative, then

$$\begin{aligned}
(zx)(xy) &= [x, y, z]z + ((zx)y)z\\
&= [x, y, z]z - [z, x, y]z + z(xy)z = z(xy)z
\end{aligned}$$

for all $x, y, z \in A$.

For (3), we apply induction, the flexible law and Moufang identity. We first claim that $x^{n+1} = xx^n$ for all $n \in \mathbb{Z}^+$. Indeed, the base case $xx^1 = x^1x = x^2$ holds; and if $x^{n+1} = xx^n$ for some $n \geq 1$, then by the flexible law: $x^{n+2} = x^{n+1}x = (xx^n)x = x(x^nx) = xx^{n+1}$. Now because $x = x^1$ we have shown that $x^{m+n} = x^mx^n$ in the base case $m = 1$. Assume for induction on m that that $x^{m+n} = x^mx^n$ for some $m \geq 1$ and $n \geq 2$ (the case $n = 1$ is obvious). Then, by the Moufang identity, $x^{m+1}x^n = (xx^m)(x^{n-1}x) = xx^{m+n-1}x = x^{m+n+1}$, as required. $\square$

Definition 1.5. An algebra A over $\mathbf{F}$ is said to be a *division algebra* if A is nonzero and $xy = 0_A \Rightarrow x = 0_A$ or $y = 0_A$ for all $x, y \in A$. $\dashv$

Remark 1.6. The term "division algebra" in Definition 1.5 comes from the following proposition, which shows that, in such an algebra, left and right division can be unambiguously performed. $\dashv$

Proposition 1.7. *Let A be an algebra over $\mathbf{F}$. Then A is a division algebra if, and only if, A is nonzero and for all $a, b \in A$, with $b \neq 0_A$, the equations $bx = a$ and $yb = a$ have unique solutions $x, y \in A$.*

Proof. ($\Rightarrow$) Fix $b \in A$, say with $b \neq 0_A$, and let $\phi : A \to A$ be the linear transformation defined by $\phi(x) = bx$. If A is a division algebra, then $\ker\phi = \{0_A\}$ and ϕ is injective. But A is finite-dimensional, so ϕ is actually bijective. Thus, the equation $bx = a$ has a unique solution. Similarly, $yb = a$ has a unique solution, by considering $y \mapsto yb$.

($\Leftarrow$) Suppose that $xy = 0_A$. If $x = 0_A$, then we're done. Otherwise, by assumption, if $x \neq 0_A$, there is a unique $y \in A$ such that $xy = 0_A$. But $x0_A = 0_A$, so $y = 0_A$. Therefore, A is a division algebra. $\square$

Corollary 1.8. *Let A be a division algebra over $\mathbf{F}$. If A is alternative, then A is unital.*

Proof. Fix $b \in A$ such that $b \neq 0_A$. By Proposition 1.7, because A is a division algebra the equation $yb = b$ has a unique solution $y = 1$. Furthermore, $1(1b) = 1b$. Since A is alternative, $1^2b = 1b$ which implies $(1^2-1)b = 0_A$ and hence $1^2 = 1$. It follows that $1(1x-x) = 1(1x)-1x = 1^2x - 1x = 0_A$. But $1 \neq 0_A$, since $b \neq 0_A$. Therefore, $1x - x = 0_A$ and $1x = x$ for all $x \in A$. Similarly, $x1 = x$ for all $x \in A$, by considering the product $(x1 - x)1$. Thus, A is unital. $\square$

Remark 1.9. In the sequel we assume $\mathbf{F} = \mathbb{R}$ and consider classes of division algebras over $\mathbb{R}$ or "real division algebras" for short. $\dashv$

The organization is as follows. In §2, we introduce the algebras of quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. Together with the real and complex numbers, these form the four classical examples of division algebras over the real numbers. Under an appropriate identification,

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}.$$

Yet, the 4-dimensional quaternions are a noncommutative field, and the 8-dimensional octonions are a nonassociative alternative algebra.

In §3, we describe the Cayley-Dickson construction of $*$-algebras (which are algebras with conjugation). When applied inductively to the real numbers, this process yields a nice proof that the quaternions and octonions are division algebras with the properties stated above.

In §4, we establish Zorn's theorem that (up to isomorphism) there are exactly four alternative real division algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. Then the classifications of the associative and the normed division algebras by Frobenius and Hurwitz follow as easy corollaries.

In §5, we outline the history of these classification theorems from Frobenius through the recent characterization of the 2-dimensional and the commutative real division algebras by Hübner and Petersson. And, in §6, we recommend articles for the reader who is interested in learning more about division algebras.

2. Quaternions and Octonions

The quaternions and octonions are alternative division algebras that extend the real and complex numbers in a natural way.

Definition 2.1. Let $\mathbb{H}$ be the 4-dimensional real algebra defined by

$$\mathbb{H} = \operatorname{span}_{\mathbb{R}}\{1, i, j, k\}$$

with identity 1 whose multiplication satisfies

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

We call $\mathbb{H}$ the algebra of *quaternions*. ⊣

Remark 2.2. The key to multiplication of quaternions is Figure 1, together with the rules

- 1 is the identity,
- i, j, k are square roots of -1.

If $x, y, z \in \{i, j, k\}$ are located consecutively in the clockwise direction (following the arrows) in Figure 1, then

$$xy = z.$$

If $x, y, z \in \{i, j, k\}$ are located consecutively in the counterclockwise direction (against the arrows) in Figure 1, then

$$xy = -z.$$

This completely determines multiplication using the distributive laws. In §3, we will demonstrate that the quaternions are a noncommutative associative division algebra. ⊣

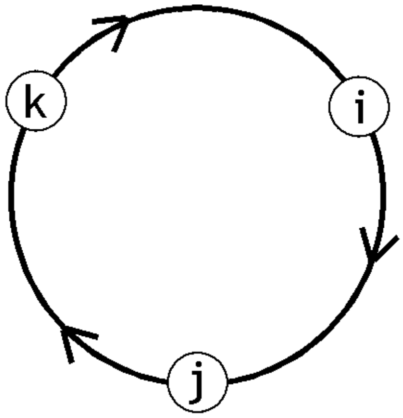

Figure 1. Quaternion 3-cycle

Example 2.3. Evaluate the quaternion product $(1+i+k)(2j-3k)$:

$$\begin{aligned}
&(1+i+k)(2j-3k) \\
&\qquad = 1(2j-3k)+i(2j-3k)+k(2j-3k) && \text{(distributive law)} \\
&\qquad = 2(1j)-3(1k)+2ij-3ik+2kj-3k^2 && \text{(distributive law)} \\
&\qquad = 2j-3k+2ij-3ik+2kj-3k^2 && \text{(1 identity)} \\
&\qquad = 2j-3k+2ij-3ik+2kj+3(1) && (k^2=-1) \\
&\qquad = 2j-3k+2k+3j-2i+3(1) && \text{(use Figure 1)} \\
&\qquad = 3(1)-2i+5j-k && \text{(collect terms)}
\end{aligned}$$

By identifying the real numbers as a subset of the quaternions using the natural inclusion $\lambda \hookrightarrow \lambda 1$ for all $\lambda \in \mathbb{R}$, we may also write

$$(1+i+k)(2j-3k) = 3-2i+5j-k.$$

As an exercise, we recommend that the reader show that

$$(2j-3k)(1+i+k) = 3+2i-j-5k,$$

in order to check the reader understands quaternion multiplication. ⊣

Definition 2.4. Let $x = a+bi+cj+dk \in \mathbb{H}$ for some $a,b,c,d \in \mathbb{R}$. The *conjugate* of x, denoted by $x \mapsto \bar{x}$, is defined by

$$\bar{x} = a-bi-cj-dk \in \mathbb{H}.$$

The *norm* of x, denoted by $x \mapsto ||x||$, is defined by

$$||x|| = \sqrt{x\bar{x}} \geq 0.$$ ⊣

Remark 2.5. If one checks that the quaternion norm is well-defined, and $x\bar{x} = \bar{x}x$ then it follows that $\mathbb{H}$ is a division algebra. Indeed, given $0 \neq x \in \mathbb{H}$, $||x|| > 0$ and $x^{-1} = ||x||^{-2}\bar{x}$ is a full inverse to x:

$$xx^{-1} = \frac{x\bar{x}}{||x||^2} = \frac{||x||^2}{||x||^2} = 1 = \frac{||x||^2}{||x||^2} = \frac{\bar{x}x}{||x||^2} = x^{-1}x$$

Hence, if $xy = 0$, then $x = 0$ or $y = x^{-1}xy = x^{-1}0 = 0$ and, thus, $\mathbb{H}$ is a division algebra. (This approach also requires that $\mathbb{H}$ is associative.) Rather than take this approach, we prefer to apply the Cayley-Dickson process described below. ⊣

Definition 2.6. Let $\mathbb{O}$ be the 8-dimensional real algebra defined by

$$\mathbb{O} = \operatorname*{span}_{\mathbb{R}}\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

with identity 1 whose multiplication satisfies

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

We call $\mathbb{O}$ the algebra of *octonions.* ⊣

Remark 2.7. While the multiplication table for the octonions may seem incomprehensible, if one thinks of the e_i as indexed by $\mathbb{Z}/7\mathbb{Z}$, then the following patterns can be observed:

- **square roots**: $e_i^2 = -1$,
- **anti-commuting**: $e_ie_j = e_k \Leftrightarrow e_je_i = -e_k$,
- **index cycling**: $e_ie_j = e_k \Leftrightarrow e_{i+1}e_{j+1} = e_{k+1}$,
- **index doubling**: $e_ie_j = e_k \Leftrightarrow e_{2i}e_{2j} = e_{2k}$.

Together with the nontrivial product $e_1e_2 = e_4$, these patterns allow us to complete the multiplication table.

As before, we prefer the use of a diagram over a multiplication table. The key to multiplication in the octonions is Figure 2, together with the rules

- 1 is the identity,
- e_1, e_2, e_3, e_4, e_5, e_6, e_7 are square roots of -1.

The Fano plane is a finite projective plane with 7 points and 7 lines, each of which is incident to 3 points. Given any pair of distinct points

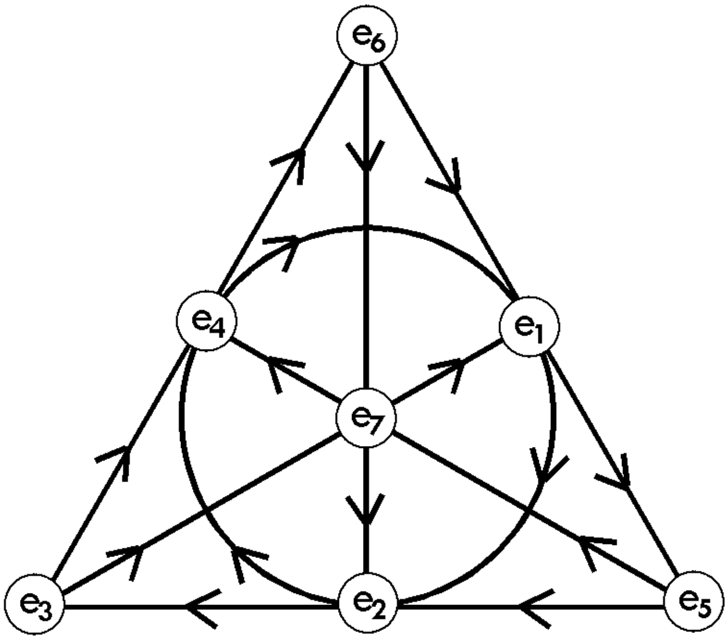

FIGURE 2. Octonion Fano plane

e_i and e_j, there is a unique e_k such that $\overline{e_i e_j e_k}$ is a line on the plane. If e_i, e_j, e_k lie consecutively in the order of the arrows, then

$$e_i e_j = e_k.$$

If e_i, e_j, e_k lie consecutively against the order of the arrows, then

$$e_i e_j = -e_k.$$

This completely determines multiplication using the distributive laws. (As an aside, we note that index doubling in the multiplication table is embedded into Figure 2 as follows: doubling the indices $e_i \mapsto e_{2i}$ corresponds to rotating Figure 2 by 120° counterclockwise about e_7.) In §3, we will show that the octonions are an alternative nonassociative division algebra. $\dashv$

Example 2.8. It is easy to see that the octonions are nonassociative. On one hand,

$$e_1(e_2 e_3) = e_1 e_5 = e_6.$$

On the other hand,

$$(e_1 e_2)e_3 = e_4 e_3 = -e_6.$$

As with the quaternions, we will identify the real numbers as a subset of the octonions using the natural inclusion map $\lambda \hookrightarrow \lambda 1$ for all $\lambda \in \mathbb{R}$.

To test understanding of octonion multiplication, we recommend that the reader show that for $x = 1 + e_1$, $y = 2e_2 + 3e_3$, $z = 4e_4 - 5e_5$,

$$x(yz) = -8 + 8e_1 - 15e_2 + 10e_3 - 15e_4 - 12e_5 + 12e_6 + 10e_7$$

$$\text{and } (xy)z = -8 + 8e_1 - 15e_2 + 10e_3 + 15e_4 + 12e_5 + 12e_6 - 10e_7. \quad \dashv$$

Definition 2.9. Let $x = a + \sum_i b_i e_i \in \mathbb{O}$ for some $a, b_1, \ldots, b_7 \in \mathbb{R}$. The *conjugate* of x, denoted by $x \mapsto \bar{x}$, is defined by

$$\bar{x} = a - \sum_{i=1}^{7} b_i e_i \in \mathbb{O}.$$

The *norm* of x, denoted by $x \mapsto ||x||$, is defined by

$$||x|| = \sqrt{x\bar{x}} \geq 0. \quad \dashv$$

Remark 2.10. If one checks that the octonion norm is well-defined, and $x\bar{x} = \bar{x}x$ then it follows that $\mathbb{O}$ is a division algebra. Indeed, given $0 \neq x \in \mathbb{O}$, $||x|| > 0$ and $x^{-1} = ||x||^{-2}\bar{x}$ is a full inverse to x:

$$xx^{-1} = \frac{x\bar{x}}{||x||^2} = \frac{||x||^2}{||x||^2} = 1 = \frac{||x||^2}{||x||^2} = \frac{\bar{x}x}{||x||^2} = x^{-1}x$$

Hence, if $xy = 0$, then $x = 0$ or $y = x^{-1}xy = x^{-1}0 = 0$ and, thus, $\mathbb{O}$ is a division algebra. (This approach also requires that $\mathbb{O}$ is alternative.) Rather than take this approach, we prefer to apply the Cayley-Dickson process which we now describe. $\dashv$

3. Cayley-Dickson Process

The Cayley-Dickson process for constructing families of algebras with "conjugation" explains why the complex numbers are commutative, but not real; the quaternions are associative, but not commutative; and the octonions are alternative, but not associative. It also explains why $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are division algebras, yet no division algebras extend the octonions, like the octonions extend the quaternions. The process mimics the construction of complex numbers as pairs of real numbers.

Definition 3.1. Let A be an algebra over $\mathbb{R}$. Then A is said to be a $*$-*algebra* if there exists a linear map called *conjugation* $* : A \to A$ (acting exponentially) such that

$$x^{**} = x, \qquad (xy)^* = y^*x^*, \qquad \text{for all } x, y \in A.$$

We call a $*$-algebra A *real*, if $x^* = x$ for all $x \in A$, and *nicely-normed*, if A is unital, $x + x^* \in \mathbb{R}$ and $x^*x = xx^* > 0$ for all nonzero $x \in A$. $\dashv$

Example 3.2. Both the real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$ are $*$-algebras under the usual complex conjugation $x^* \mapsto \bar{x}$ for all $x \in \mathbb{C}$. Both $*$-algebras are nicely-normed since both have an identity and

$$x + x^* = x + \bar{x} = 2\operatorname{Re} x \in \mathbb{R} \quad \text{and} \quad x^*x = xx^* = x\bar{x} = |x|^2 \geq 0$$

for all $x \in \mathbb{C}$. Moreover, $\mathbb{R}$ is real since $x^* = \bar{x} = x$ for all $x \in \mathbb{R}$; yet, $\mathbb{C}$ is not real since $i^* = \bar{i} = -i \neq i$ and $i \in \mathbb{C}$. ⊣

Definition 3.3. Let A be a nicely-normed $*$-algebra, and let $x \in A$. The *norm* of x, denoted by $x \mapsto ||x||$, is defined by

$$||x|| = \sqrt{xx^*} \geq 0.$$

If $x \neq 0$, the *inverse* of x, denoted by $x \mapsto x^{-1}$, is defined by

$$x^{-1} = ||x||^{-2}x^*. \qquad ⊣$$

Proposition 3.4. *The norm and inverse defined above are well-defined.*

Proof. Since the $*$-algebra A is nicely-normed, $xx^* \geq 0$ for all $x \in A$, with equality if and only if $x = 0$. Thus, $\sqrt{xx^*} \geq 0$ exists and the norm is well-defined for all $x \in A$. If $x \neq 0$, it follows that

$$xx^{-1} = \frac{xx^*}{||x||^2} = \frac{||x||^2}{||x||^2} = 1 = \frac{||x||^2}{||x||^2} = \frac{x^*x}{||x||^2} = x^{-1}x$$

where xx^* and x^*x commute again since A is nicely-normed. Therefore, x^{-1} is the full inverse of x and well-defined for all nonzero $x \in A$. □

Corollary 3.5. *Let A be a nicely-normed $*$-algebra. If A is alternative, then A is a division algebra.*

Proof. Let $xy = 0$ for some $x, y \in A$, and suppose that $x \neq 0$. To prove A is a division algebra, we must show $y = 0$. Because A is alternative, by the Moufang identity (Lemma 1.4.2),

$$yx^{-1} = 1(yx^{-1}) = (x^{-1}x)(yx^{-1}) = x^{-1}(xy)x^{-1} = x^{-1}0x^{-1} = 0$$

where $x^{-1} = ||x||^{-2}x^*$ is well-defined by the previous proposition. Hence, $yx^* = 0$, which implies $xy^* = (yx^*)^* = 0^* = 0$. Thus,

$$x(y + y^*) = xy + xy^* = 0 + 0 = 0. \tag{$\diamondsuit$}$$

Since A is nicely-normed, $y + y^* \in \mathbb{R}$; but $x \neq 0$, so $y + y^* = 0$ by ($\diamondsuit$), or equivalently, $y = -y^*$. Therefore, again since A is alternative,

$$-||y||^2x = x(-yy^*) = x(yy) = (xy)y = 0y = 0.$$

We conclude that $||y|| = 0$ which occurs if and only if $y = 0$. □

Definition 3.6. Let A be a $*$-algebra. The *Cayley-Dickson extension* of A, which we denote by A', is the $*$-algebra $A \times A$ satisfying

addition:	$(a,b)+(c,d)=(a+c,b+d)$
scalar product:	$\lambda(a,b)=(\lambda a,\lambda b)$
multiplication:	$(a,b)(c,d)=(ac-db^*,a^*d+cb)$
conjugation:	$(a,b)^*=(a^*,-b)$

for all $a,b,c,d \in A$ and $\lambda \in \mathbb{R}$. $\dashv$

Example 3.7. Up to isomorphism: $\mathbb{R}'=\mathbb{C}$, $\mathbb{C}'=\mathbb{H}$, $\mathbb{H}'=\mathbb{O}$.

Clearly $\mathbb{R}'=\mathbb{C}$ with $(0,1)=i$. Since $\mathbb{R}$ is real, $x^*=x$ for all $x\in\mathbb{R}$; hence, the relations for multiplication and conjugation in $\mathbb{R}'$ satisfy

$$(a,b)(c,d)=(ac-db,ad+cb) \quad \text{and} \quad (a,b)^*=(a,-b)$$

for all $a,b,c,d\in\mathbb{R}$ where in $\mathbb{C}$ these satisfy

$$(a+bi)(c+di)=ac-db+(ad+cb)i \quad \text{and} \quad \overline{a+bi}=a-bi.$$

The reader can similarly check the isomorphisms for $\mathbb{C}'=\mathbb{H}$, $\mathbb{H}'=\mathbb{O}$. For the quaternions, make the identification:

$$i=(i,0), \qquad j=(0,1), \qquad \text{and} \qquad k=(0,-i).$$

For the octonions, make the identification:

$$\begin{aligned} &e_1=(i,0), && e_2=(j,0), && e_3=(0,1), && e_4=(k,0),\\ &e_5=(0,-j), && e_6=(0,k), && \text{and} && e_7=(0,-i). \end{aligned}$$ $\dashv$

Theorem 3.8 (Properties of Extensions)**.** *Let A be a $*$-algebra. Then*

(1) *A' is never real (unless trivially $A=0$).*
(2) *A is real (and thus commutative) $\Leftrightarrow$ A' is commutative.*
(3) *A is commutative and associative $\Leftrightarrow$ A' is associative.*
(4) *A is associative and nicely-normed $\Leftrightarrow$ A' is alternative and nicely-normed.*
(5) *A is nicely-normed $\Leftrightarrow$ A' is nicely-normed.*

Proof. For (1), choose $b\in A$ such that $b\neq 0$. Then $(0,b)\in A'$, but

$$(0,b)^*=(0,-b)=-(0,b)\neq(0,b).$$

Thus, A' is not real.

For (2), suppose first that A is real. Then A is also commutative since for any $a,b\in A$, $ab=(b^*a^*)^*=(ba)^*=ba$. Hence, A' is commutative, since for any $(a,b),(c,d)\in A'$:

$$(a,b)(c,d)=(ac-db^*,a^*d+cb)=(ca-bd^*,d^*a+bc)=(c,d)(a,b).$$

Conversely, suppose that A' is commutative and let $a \in A$. Then

$$(a^*, 0) = (0, a)(0, -1) = (0, -1)(0, a) = (a, 0).$$

Hence, $a^* = a$ for all $a \in A$ and A is real.

For (3), if A is commutative and associative, then A' is associative. Indeed, for all $(a, b), (c, d), (e, f) \in A'$, by the assumed properties of A:

$$\begin{aligned}
&(a, b)((c, d)(e, f)) \\
&\qquad = (a, b)(ce - fd^*, c^*f + ed) \\
&\qquad = (a(ce - fd^*) - (c^*f + ed)b^*, a^*(c^*f + ed) + (ce - fd^*)b) \\
&\qquad = (ace - afd^* - c^*fb^* - edb^*, a^*c^*f + a^*ed + ceb - fd^*b) \\
&\qquad = (ace - db^*e - fd^*a - fb^*c^*, c^*a^*f - bd^*f + ea^*d + ecb) \\
&\qquad = ((ac - db^*)e - f(a^*d + cb)^*, (ac - db^*)^*f + e(a^*d + cb)) \\
&\qquad = (ac - db^*, a^*d + cb)(e, f) \\
&= ((a, b)(c, d))(e, f).
\end{aligned}$$

On the other hand, suppose that A' is associative and let $a, b, c \in A$. Then A is commutative, since

$$\begin{aligned}
(0, ab) = (a^*, 0)(0, b) &= ((0, a)(0, -1))(0, b) \\
&= (0, a)((0, -1)(0, b)) = (0, a)(b, 0) = (0, ba).
\end{aligned}$$

Also A is associative, since

$$\begin{aligned}
(a(bc), 0) = (a, 0)(bc, 0) &= (a, 0)((b, 0)(c, 0)) \\
&= ((a, 0)(b, 0))(c, 0) = (ab, 0)(c, 0) = ((ab)c, 0).
\end{aligned}$$

For (5), in the forward direction, suppose that A is nicely-normed. Let $(a, b) \in A'$. Then $(a, b) + (a, b)^* = (a, b) + (a^*, -b) = (a + a^*, 0) \in \mathbb{R}$, since A is nicely-normed and $a + a^* \in \mathbb{R}$. Also, since A is nicely-normed, if $a \neq 0$ or $b \neq 0$ (so that $(a, b) \neq 0$), then

$$\begin{aligned}
(a, b)(a, b)^* &= (a, b)(a^*, -b) \\
&= (aa^* + bb^*, -a^*b + a^*b) \\
&= (aa^* + bb^*, 0) = (aa^*, 0) + (bb^*, 0) > 0 + 0 = 0. \\
&= (a^*a + b^*b, ab - ab) \\
(a, b)^*(a, b) &= (a^*, -b)(a, b)
\end{aligned}$$

Hence, A' is nicely-normed. In the reverse, assume A' is nicely-normed and let $a \in A$. Then $a + a^* \equiv (a, 0) + (a^*, 0) = (a, 0) + (a, 0)^* \in \mathbb{R}$. Similarly, if also $a \neq 0$, then $aa^* \equiv (a, 0)(a^*, 0) = (a, 0)(a, 0)^* > 0$ and $a^*a \equiv (a^*, 0)(a, 0) = (a, 0)^*(a, 0) > 0$. Thus, A is nicely-normed.

For (4), suppose first that A is associative and nicely-normed. Then A' is nicely-normed by (5) and it remains to show that A' is alternative. Let $(a,b),(c,d) \in A'$. Then, since A is associative and nicely-normed, the left alternative law holds:

$$\begin{aligned}
&(a,b)((a,b)(c,d)) \\
&\qquad = (a,b)(ac - db^*, a^*d + cb) \\
&\qquad = (a(ac - db^*) - (a^*d + cb)b^*, a^*(a^*d + cb) + (ac - db^*)b) \\
&\qquad = (aac - adb^* - a^*db^* - cbb^*, a^*a^*d + a^*cb + aca - db^*a) \\
&\qquad = (aac - bb^*c - db^*a - db^*a^*, a^*a^*d - b^*bd + ca^*b + cab) \\
&\qquad = ((aa - bb^*)c - d(a^*b + ab)^*, (aa - bb^*)^*d + c(a^*b + ab)) \\
&\qquad = (aa - bb^*, a^*b + ab)(c,d) \\
&= ((a,b)(a,b))(c,d).
\end{aligned}$$

Similarly, the right alternative law holds:

$$\begin{aligned}
&(a,b)((c,d)(c,d)) \\
&\qquad = (a,b)(cc - dd^*, c^*d + cd) \\
&\qquad = (a(cc - dd^*) - (c^*d + cd)b^*, a^*(c^*d + cd) + (cc - dd^*)b) \\
&\qquad = (acc - add^* - c^*db^* + cdb^*, a^*c^*d + a^*cd + ccb - dd^*b) \\
&\qquad = (acc - db^*c - dd^*a + db^*c^*, c^*a^*d - bd^*d + ca^*d + ccb) \\
&\qquad = ((ac - db^*)c - d(a^*d + cb)^*, (ac - db^*)^*d + c(a^*d + cb)) \\
&\qquad = (ac - db^*, a^*d + cb)(c,d) \\
&= ((a,b)(c,d))(c,d).
\end{aligned}$$

Conversely, now suppose that A' is alternative and nicely-normed. Then A is nicely-normed by (5); we must show that A is associative. Let $a,b,c \in A$. We leave it as a challenge for the reader to show $a(bc) = (ab)c$. □

Corollary 3.9.

$\mathbb{R}$ *is a real commutative associative nicely-normed* $*$*-algebra* $\Rightarrow$
$\mathbb{C}$ *is a commutative associative nicely-normed* $*$*-algebra* $\Rightarrow$
$\mathbb{H}$ *is an associative nicely-normed* $*$*-algebra* $\Rightarrow$
$\mathbb{O}$ *is an alternative nicely-normed* $*$*-algebra* $\Rightarrow$

and therefore $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ *and* $\mathbb{O}$ *are division algebras.*

Remark 3.10. Given any nonzero $*$-algebra A, the Cayley-Dickson extension A' is clearly a $*$-algebra with twice the dimension of A. Hence, with initial input $A = \mathbb{R}$, taking Cayley-Dickson extensions inductively yields a nested sequence of real algebras with conjugation beginning with the 2-dimensional complex numbers, the 4-dimensional quaternions, and the 8-dimensional octonions. Yet, as we illustrated by Theorem 3.8, each extension loses a property of its predecessor: the complex numbers are no longer real, the quaternions are no longer commutative, the octonions are no longer associative, and the *sedonions* $\mathbb{O}'$ are no longer alternative (but are power-associative). Nevertheless, nonzero sedonions have inverses by Theorem 3.8.5 and Proposition 3.4. But we cannot apply Corollary 3.5 to $\mathbb{O}'$. In fact, the sedonions have zero divisors! Therefore, only the first four algebras in the sequence $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{O}', \mathbb{O}'', \ldots$ are division algebras. This fact is a special case of the (1,2,4,8)-Theorem discussed in §5. See [12] and [15] for references on the sedonions. $\dashv$

4. Alternative Division Algebras

Our proof of the classification of alternative real division algebras follows the presentation given by Oneto in [16]. We then obtain the classifications of associative and normed division algebras as corollaries.

Throughout this section, we let D be a fixed alternative real division algebra; recall that D is unital, by Corollary 1.8, say with identity 1. Let R be the subalgebra of D induced by the inclusion $\lambda \hookrightarrow \lambda 1$ for all $\lambda \in \mathbb{R}$ so that R is naturally isomorphic to $\mathbb{R}$.

Lemma 4.1. *If $x \in D$, then $x^2 \in Rx + R$.*

Proof. Let $R[X]$ denote the polynomial algebra with indeterminant X and coefficients in R. Because D is power-associative (Lemma 1.4.3), the specialization $R[X] \to D$ given by $X \mapsto x$ extends to a morphism of algebras. The set of powers $\{1, x, x^2, x^3, \ldots\}$ of an element $x \in D$ is linearly dependent, since D has finite dimension. Since a polynomial in $R[X] \cong \mathbb{R}[X]$ is a product of polynomials of degree one or two, and since D has no zero divisors, we conclude that x satisfies a linear or quadratic equation with coefficients in R. In either case, we can write $x^2 = ax + b$ for some coefficients $a, b \in R$. $\square$

Lemma 4.2. *If $D \neq R$, then there exists $i \in D$ such that $i^2 = -1$, and $C := R + Ri$ is isomorphic to $\mathbb{C}$. Furthermore, $C = \{x \in D : xi = ix\}$, and setting $C^- := \{x \in D : xi = -ix\}$ we have $D = C \oplus C^-$.*

Proof. Pick $x \in D \backslash R$. By Lemma 1, $x^2 = ax + b$ for some $a, b \in R$ and so $(x - \frac{a}{2})^2 \in R$. Since $x \notin R$, we must have $(x - \frac{a}{2})^2 = -c^2$ for some $c \in R$. Setting $i := c^{-1}(x - \frac{a}{2})$, we have found an $i \in D$ with $i^2 = -1$. Define $C := R + Ri$, which is isomorphic to $\mathbb{C}$.

It is clear that $R + Ri \subseteq \{x \in D : xi = ix\}$. To show equality, suppose that $x \in D$ such that $xi = ix$. If $x \in R$, then $x \in R + Ri$

trivially. Otherwise, if $x \notin R$, then (arguing as before) there exist $b, d \in R$ with $(x - \frac{b}{2})^2 = -d^2 = (id)^2$. But since $xi = ix$,

$$\left(x - \frac{b}{2}\right)^2 - (id)^2 = \left(x - \frac{b}{2} + id\right)\left(x - \frac{b}{2} - id\right) = 0$$

and by the division property $x = \frac{b}{2} \pm id$, so $x \in R + Ri$. Hence, we have shown $C = \{x \in D : xi = ix\}$.

Let $C^- := \{x \in D : xi = -ix\}$. Obviously C^- is a subspace of D and $C \cap C^- = 0$. To show $D = C \oplus C^-$, it remains to prove that $D = C + C^-$. But this is a consequence of the identity:

$$x = \frac{1}{2}(x - ixi) + \frac{1}{2}(x + ixi) \tag{$\clubsuit$}$$

where $x - ixi \in C$ and $x + ixi \in C^-$ by the alternative laws and by the flexible law (Lemma 1.4.1). $\square$

Lemma 4.3. *If $x, y \in D$ anticommute, then x^2 and y commute.*

Proof. Let $x, y \in D$ such that $xy = -yx$. We apply the alternative and flexible laws: $x^2y = x(xy) = -x(yx) = -(xy)x = (yx)x = yx^2$. $\square$

Lemma 4.4. *If $x, y \in D$ anticommute, then $x(yz) = -y(xz)$ and $(zx)y = -(zy)x$ for all $z \in D$.*

Proof. For all $x, y, z \in D$, $[x, y, z] + [y, x, z] = 0$, since D is alternative. Since also $xy + yx = 0$:

$$0 = [x, y, z] + [y, x, z] + (xy + yx)z = x(yz) + y(xz).$$

Thus, $x(yz) = -y(xz)$. Similarly, $(zx)y = -(zy)x$. $\square$

Lemma 4.5. *If $D \not\subseteq C$, then there exists $j \in C^-$ such that $j^2 = -1$, and $H := C + Cj$ is isomorphic to $\mathbb{H}$. Furthermore, writing $k := ij$, $H = \{x \in D : xk = (xi)j\}$, and setting $H^- := \{x \in D : xk = -(xi)j\}$ we have $D = H \oplus H^-$.*

Proof. Pick $x \in C^-$ such that $x \neq 0$. By Lemma 4.3, Lemma 4.2 and Lemma 4.1 we have $x \in C \cap (R + Rx)$ but $Rx \subseteq C^-$ and since $D = C \oplus C^-$, $x^2 \in R$. If $x^2 > 0$ then $x \in R$, which contradicts $D = C \oplus C^-$. Hence, $x^2 = -c^2$ for some nonzero $c \in R$. Setting $j := c^{-1}x$, we obtain $j^2 = -1$ and $ji = -ij$. Writing $k := ij$, we can deduce the quaternion identities:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

For example, by the Moufang identity, $k^2 = (ij)(ji) = -(ij)(ji) = -ij^2i = i^2 = -1$. Define $H := C + Cj$, which is isomorphic to $\mathbb{H}$.

Since H is associative, $H \subseteq \{x \in D : xk = (xi)j\}$. For this proof only, we abbreviate $K := \{x \in D : xk = (xi)j\}$. To prove the reverse inclusion and establish equality, we must first establish:

$$K = C \oplus (C^- \cap K). \tag{$\spadesuit$}$$

By Lemma 4.2 and ($\clubsuit$), it suffices to verify: if $x \in K$, then $x + ixi \in K$. From Lemma 4.4 and the Moufang identity we have:

$$(x + ixi)k = xk - i(xk)i = xk - (ix)(ki) = xk - (ix)j.$$

By the right alternative law:

$$((x + ixi)i)j = (xi)j - (ix)j.$$

But if $x \in K$, then $xk = (xi)j$ and so $x + ixi \in H$ and ($\spadesuit$) holds.

Right multiplication by j defines a linear transformation $T(x) \mapsto xj$ that maps K into itself. In fact, if $x \in K$, then by Lemma 4.4:

$$\begin{aligned} (xj)k &= -(xk)j = -((xi)j)j = xi \\ ((xj)i)j &= -((xj)j)i = xi. \end{aligned}$$

Hence $xj \in K$. Also, T interchanges C and $C^- \cap K$. Indeed, we have:

$$(xj)i = -(xi)j = -(ix)j = -i(xj)$$

with the last equality since $0 = [x, i, j] = -[i, x, j] = -i(xj) + (ix)j$. Hence, if $x \in C$, then $xj \in C^- \cap K$. Similarly, if $x \in C^- \cap K$, then $xj \in C$. But T is an automorphism (its inverse is $T^{-1}(x) \mapsto -xj$). Thus, $\dim(C' \cap K) = \dim(C) = 2$ and by ($\spadesuit$) K is a 4-dimensional subspace. Thus, because $H \subseteq K$, $H = K = \{x \in D : xk = (xi)j\}$.

Define $H^- = \{x \in D : xk = -(xi)j\}$, which is obviously a subspace such that $H \cap H^- = \{0\}$. To show that $D = H \oplus H^-$, it remains to prove $D = H + H^-$. We use the identity

$$x = \frac{1}{2}(x - ((xi)j)k) + \frac{1}{2}(x + ((xi)j)k).$$

By this identity, it suffices to verify that $x - ((xi)j)k \in H$ and $x + ((xi)j)k \in H^-$. For the former, by Lemma 4.4 and alternative laws:

$$\begin{aligned} (x + ((xi)j)k)k &= xk - xij \\ (x + ((xi)j)k)ij &= xij + xi^2jkj = xij - ((xi)j)k = xij - xk \end{aligned}$$

so $x + ((xi)j)k \in H$. The latter inclusion holds similarly. Therefore, $D = H \oplus H^-$, as desired. $\square$

Lemma 4.6. *If $x \in H^-$, then x anticommutes with i, j and k.*

Proof. Let $x \in X^-$. By definition, $(xi)j = -k$. First, we show x anticommutes with k. We have $xk = \frac{1}{2}xk + \frac{1}{2}xk = \frac{1}{2}(ij) - \frac{1}{2}(xi)j = \frac{1}{2}[x, i, j]$. But $xk = -(xi)j$ implies $x = [(xi)j]k$. By the Moufang identity:

$$kx = [k(xi)](jk) = [k(xi)]i = [(ij)(xi)]i = \{[i(jx)]i\}i = -i(jx)$$

and so $kx = -\frac{1}{2}[i, j, x] = -\frac{1}{2}[x, i, j] = -xk$.

Second, we show x anticommutes with j. By Lemma 4.4, we have $x = -[(xi)k]j$ (since $x \in H^-$). By the Moufang identity:

$$jx = -[j(xi)](kj) = [j(xi)]i = -[(ik)(xi)]i = -[i(kx)i]i = i(kx).$$

Hence, $jx = \frac{1}{2}[i, k, x] = \frac{1}{2}[x, i, k] = -\frac{1}{2}xj - \frac{1}{2}(xi)k = -xj$.

Finally, x anticommutes with i, since x anticommutes with j and k: $xi = (xk)j = -(kx)j = (kj)x = -ix$. □

Lemma 4.7. *If $D \not\subseteq H$, then there exists $h \in H^-$ such that $h^2 = -1$.*

Proof. Let $x \in H^-$, say with $x \neq 0$. By Lemma 4.6, $x \in C^-$ and from Lemma 4.3, Lemma 4.2 and Lemma 4.1 we find: $x^2 \in C \cap (R + Rx) = R$. But $x \notin R$ (since $x \in H^-$), thus $x^2 = -c^2$ for some nonzero $c \in R$. Setting $h = c^{-1}x$, the claim follows. □

Theorem 4.8 (Zorn)**.** *If A is an alternative real division algebra, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Proof. Let us apply the framework we developed above with $D = A$. If $D = R$, then D is isomorphic to $\mathbb{R}$, by the paragraph proceeding Lemma 4.1. Otherwise, by Lemma 4.2, D contains an algebra C which is isomorphic to $\mathbb{C}$. If $D = C$, we're done. Otherwise, by Lemma 4.5, D contains an algebra H which is isomorphic to $\mathbb{H}$. Again, if $D = H$, we're done. Suppose now that $D \not\subseteq H$.

By Lemma 4.5 and Lemma 4.7, $D = H \oplus H^-$ and there exists $h \in H^-$ such that $h^2 = -1$. The mapping $T(x) \mapsto xh$ defines a linear automorphism of D (since D is alternative, T has an inverse $T^{-1}(x) \mapsto -xh$). Observe that T interchanges H and H^-. If $x \in H$, then by Lemma 4.4 and Lemma 4.6,

$$T(x)k = (xh)k = -(xk)h = -((xi)j)h = -((xh)i)j.$$

Hence, if $x \in H$, then $T(x) \in H^-$. Similarly, if $x \in H^-$, then $T(x) \in H$. Thus, $\dim H^- = \dim H = 4$ (since $H \cong \mathbb{H}$), and it follows that

$$\dim(D) = \dim(H \oplus H^-) = \dim(H) + \dim(H^-) = 8.$$

It also follows that $H^- = Hh$ and $\{h, ih, jh, kh\}$ is a basis of H^-. Therefore, $\{1, i, j, k, h, ih, jh, kh\}$ is a basis for D. It now remains to compute the multiplication table for D. For instance,

$$(ih)(kh) = -(hi)(kh) = -h(ik)h = hjh = (-jh)h = j,$$
$$(ih)(ih) = -(ih)(hi) = -ih^2i = i^2 = -1.$$

A complete multiplication table for D is given by:

	i	j	ih	k	kh	jh	$-h$
i	-1	k	$-h$	$-j$	jh	$-kh$	$-ih$
j	$-k$	-1	kh	i	$-ih$	$-h$	$-jh$
ih	h	$-kh$	-1	jh	j	$-k$	i
k	j	$-i$	$-jh$	-1	$-h$	ih	$-kh$
kh	$-jh$	ih	$-j$	h	-1	i	k
jh	kh	h	k	$-ih$	$-i$	-1	j
$-h$	ih	jh	$-i$	kh	$-k$	$-j$	-1

By our chosen ordering of the table (we recommend that the reader relabel Figure 2), we clearly have $D \cong \mathbb{O}$, given explicitly by $e_1 \mapsto i$, $e_2 \mapsto j$, $e_3 \mapsto ih$, $e_4 \mapsto k$, $e_5 \mapsto kh$, $e_6 \mapsto jh$ and $e_7 \mapsto -h$.

We have thus shown that an alternative real division algebra A is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. □

Corollary 4.9 (Frobenius)**.** *If A is an associative real division algebra, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Proof. Since A is associative, A is alternative. Hence, by Theorem 4.8, A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. But $\mathbb{O}$ is not associative, which is preserved under isomorphism. Thus, A is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. □

Definition 4.10. Let A be a real division algebra with identity 1. Then A is said to be *normed* if there is an inner product $(\cdot,\cdot)$ on A such that

$$(\heartsuit) \qquad (xy, xy) = (x,x)(y,y) \quad \text{for all } x, y \in A. \qquad \dashv$$

Corollary 4.11 (Hurwitz)**.** *If A is a normed real division algebra, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Proof. By Theorem 4.8, it suffices to show that A is alternative. Write $x + x'$ in place of x in $(\heartsuit)$ and use linearity of the inner product to find

$$(xy, xy) + 2(xy, x'y) + (x'y, x'y) = ((x,x) + 2(x,x') + (x',x'))(y,y).$$

Apply $(\heartsuit)$ to the left hand side and cancel like terms to yield

$$(\heartsuit_1) \qquad (xy, x'y) = (x,x')(y,y).$$

Next, write $y + y'$ in place of y and use linearity again to obtain

$$\begin{aligned}(x'y, xy) + (x'y', xy) + (x'y, xy') + (x'y', xy')\\ = (x, x')((y, y) + 2(y, y') + (y', y')).\end{aligned}$$

Apply $(\heartsuit_1)$ to the left hand side and cancel like terms to find

$$(\heartsuit_2) \qquad (xy, x'y') + (xy', x'y) = 2(x, x')(y, y') \quad \text{for all } x, x', y, y' \in A.$$

Applying $(\heartsuit_2)$ with

(3) $x' = z$ and $y' = y$, (4) $x' = x$ and $y' = z$,
(5) $x' = z$ and $y' = 1$, (6) $x' = 1$ and $y' = z$,

we obtain, respectively,

$$(\heartsuit_3) \qquad (xy, zy) = (x, z)(y, y) \quad \text{for all } x, y, z \in A,$$

$$(\heartsuit_4) \qquad (xy, xz) = (x, x)(y, z) \quad \text{for all } x, y, z \in A,$$

$$(\heartsuit_5) \qquad (xy, z) + (x, zy) = 2(y, 1)(x, z) \quad \text{for all } x, y, z \in A,$$

$$(\heartsuit_6) \qquad (xy, z) + (xz, y) = 2(x, 1)(y, z) \quad \text{for all } x, y, z \in A.$$

Now, substitute (xy) for x in $(\heartsuit_5)$ and substitute (xz) for z in $(\heartsuit_6)$. From $(\heartsuit_3)$ and $(\heartsuit_4)$ it follows that

$$\begin{aligned}((xy)y, z) + (y, y)(x, z) = 2(y, 1)(xy, z) \quad \text{for all } x, y, z \in A,\\ (x(xz), y) + (x, x)(y, z) = 2(x, 1)(y, xz) \quad \text{for all } x, y, z \in A.\end{aligned}$$

Hence,

$$\begin{aligned}((xy)y + (y, y)x - 2(y, 1)xy, z) = 0 \quad \text{for all } z,\\ (x(xz) + (x, x)z - 2(x, 1)xz, y) = 0 \quad \text{for all } y.\end{aligned}$$

Thus,

$$(\heartsuit_7) \qquad (xy)y = 2(y, 1)xy - (y, y)x \quad \text{for all } x, y \in A,$$

$$(\heartsuit_8) \qquad x(xy) = 2(x, 1)xy - (x, x)y \quad \text{for all } x, y \in A.$$

Putting $y = 1$ in $(\heartsuit_8)$, we obtain

$$(\heartsuit_9) \qquad x^2 = 2(x, 1)x - (x, x) \quad \text{for all } x \in A.$$

Finally, right multiply $(\heartsuit_9)$ by y, then compare with $(\heartsuit_8)$:

$$(x^2)y = x(xy).$$

And left multiply $y^2 = 2(y, 1)y - (y, y)$ by x, then compare with $(\heartsuit_7)$:

$$x(y^2) = (xy)y.$$

Therefore, A is alternative, as desired. □

Remark 4.12. The careful reader will notice that we have not shown $\mathbb{H}$ and $\mathbb{O}$ are normed algebras. Given $A = \mathbb{H}$ or $\mathbb{O}$, define an inner product on A by $(\cdot, \cdot) : A \times A \to \mathbb{R}$ by $(x, y) = x\bar{y}$ for all $x, y \in A$. Then the reader can show that $(xy, xy) = (x, x)(y, y)$ for all $x, y \in A$ by direct calculation or by appealing to the framework for nicely-normed $*$-algebras developed in §3. $\dashv$

5. Historical Remarks

The quaternions were the first noncommutative structure studied. Fascinated by his construction of the complex numbers as a pair of real numbers, Sir William Rowan Hamilton looked for a system of triples that satisfied the nice properties of the real and the complex numbers. But since there are no 3-dimensional real division algebras, Hamilton's attempts failed. Fortunately, Hamilton did not stop searching...

As the story goes, in October 1843, Hamilton was out walking with his wife in Dublin, when he discovered the quaternions. He later wrote, "That is to say, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j, k; exactly such as I have used them every since." Then, in an act of mathematical vandalism, he carved his equations into the Brougham Bridge: $i^2 = j^2 = k^2 = ijk = -1$. The next day, Hamilton wrote to his friend John T. Graves about his discovery. Two months later, in December 1843, Graves replied with a description of his "octaves"—the octonions. Thus, the discovery of the quaternions launched the development of hypercomplex algebras.

The classification of real division algebras began in 1878, when Georg Frobenius [7] showed that (up to isomorphism) there are exactly three such algebras which are associative: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, and the quaternions $\mathbb{H}$. In 1898, Adolph Hurwitz [10] showed secondly that the octonions are the only nonassociative real division algebra with a multiplicative norm. Then, in 1930, Max Zorn [19] generalized the results of Frobenius and Hurwitz, proving that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only alternative real division algebras.

In 1940, topologist Heinz Hopf [9] showed that (as vector spaces) division algebras over the real numbers necessarily have dimension 2^n, for some integer $n \geq 0$. Of course, the four classic examples show the existence of real division algebras in dimensions 1, 2, 4 and 8. In 1958, Rauol Bott and John Milnor [3] and Michel Kervaire [13] independently proved the deep result that real division algebras in higher dimensions do not exist:

(1,2,4,8)-Theorem. *Let A be a division algebra over the real numbers. Then A has dimension either 1, 2, 4 or 8.*

To date, the (1,2,4,8)-Theorem has avoided a purely algebraic proof. Indeed the Bott-Milnor-Kervaire proofs of the (1,2,4,8)-Theorem are obtained as corollaries to a result on a topological property, called the parallelizability of the n-sphere.

It is easy to show that any 1-dimensional real division algebra is isomorphic to the real numbers. Indeed, if A is any such an algebra, then $A = \mathbb{R}a$ for some nonzero $a \in A$. Since A is a division algebra the equation $xa = a$ has a unique solution μa, for some nonzero $\mu \in \mathbb{R}$. Then μa is an identity for A: because $\mu a^2 = a$, for any $\lambda a \in A$, $(\mu a)(\lambda a) = (\lambda a)(\mu a) = (\lambda\mu)a^2 = \lambda(\mu a^2) = \lambda a$. Therefore, since A has dimension 1, the map $\lambda \hookrightarrow \lambda(\mu a)$ for all $\lambda \in \mathbb{R}$ gives the isomorphism.

In higher dimensions, however, the picture is not as simple. Consider:

Example 5.1. The real division algebra $(\mathbb{C}, \tau)$ with multiplication

$$\tau(x, y) = \bar{x}\bar{y} \quad \text{for all } x, y \in \mathbb{C}$$

is commutative, nonalternative and nonunital. ⊣

The classification of 2-dimensional real division algebras began in 1983 when Althoen and Kugler [1] gave neccessary conditions on the multiplications tables of real division algebras in dimension 2. Sufficient conditions were provided in 1985 by Burdugan [4] (without proof) and also in 1998 by Gottschling [8]. At the heart of each of these papers is a theorem which states a 2-dimensional real division algebra has either 1, 2 or 3 idempotents (nonzero elements e such that $e^2 = e$).

Another famous theorem of Hopf states that any commutative real division algebra has dimension at most 2. The classification of all the commutative division algebras was completed in 1983 by Kantor and Solodovnikov in [14]. In view of the remarks above, a commutative real division algebra is isomorphic to either the real numbers or a subset of the 2-dimensional real division algebras classified by Althoen-Kugler.

In 2004, Hübner and Peterssen [11] published a different approach to the classification of 2-dimensional (and also commutative) real division algebras. Instead of considering the number of idempotents and placing constraints on the multiplication tables of the algebra, they write the isomorphism classes in the form $\mathbb{C}^{(f,g)}$ where $(f, g) \in \mathrm{GL}(\mathbb{R}^2) \times \mathrm{GL}(\mathbb{R}^2)$ with multiplication m satisfying $m(x, y) = f(x)g(y)$ for all $x, y \in \mathbb{C}$. For instance, $\mathbb{C}^{(\tau,\tau)}$ is given by Example 5.1 above, where we may think of τ as a reflection in the x-axis. Hübner and Peterssen show there are four families of endomorphism pairs (f, g) which yield division algebras.

6. Suggested Reading

Our interest in division algebras developed after reading John Baez' excellent survey [2] on the octonions. With the classification of the 4- and the 8-dimensional real division algebras still an incomplete task, this area is wide open for research. Of particular interest, we would like to see a purely algebraic proof of the (1,2,4,8)-Theorem. See [5], which discusses the dimensions of division algebras over arbitrary fields, and [11], for details on the Hübner-Petersson classification of the 2-dimensional and commutative real division algebras. Finally, we would recommend [6] for a good general reference (very readable!) on the quaternions, octonions and real division algebras.

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