LOJASIEWICZ INEQUALITIES AND GENERIC SMOOTHNESS OF NODAL SETS OF SOLUTIONS TO ELLIPTIC PDE

MATTHEW BADGER, MAX ENGELSTEIN, AND TATIANA TORO

Abstract. In this article, we prove that for a broad class of second order elliptic PDEs, including the Laplacian, the zero sets of solutions to the Dirichlet problem are smooth for “generic” $L^2$ data. When the zero set of a solution (e.g. a harmonic function) contains a singularity, this means that we can find an arbitrarily small perturbation of the boundary data so that the zero set of the perturbed solution is smooth throughout a prescribed neighborhood of the former singularity. Furthermore, we can take the perturbation to be “mean zero” for which there are additional technical difficulties to ensure that we do not introduce new singularities in the process of eliminating the original ones. Of independent interest, in order to prove the main theorem, we establish an effective version of the Lojasiewicz gradient inequality with uniform constants in the class of solutions with bounded frequency.

1. Introduction

In this paper, we prove that nodal sets $\{u = 0\}$ of solutions to the Dirichlet problem for elliptic PDEs are smooth for “generic” $L^2$ data. Our key tool is an “effective” Lojasiewicz gradient inequality (Theorem 1.3), where the associated constants and exponents remain uniform over a given family of solutions to a broad class of second order elliptic PDEs. One advantage of this approach, in contrast to recent deep work on generic smoothness in non-linear contexts (e.g. [FROS20, CCMS20]), is that it allows for perturbations which are not “signed” or “one-sided”. More precisely, we show that generic smoothness holds in the class of Dirichlet data with mean zero, when the equation is purely second order, and for solutions to PDEs with zeroth order terms:

Theorem 1.1 (Main Theorem, Simplified). Let $Lu \equiv -\text{div}(A \nabla u) + cu$, where $A$ is a Lipschitz continuous function with values in uniformly elliptic $n \times n$ symmetric real-valued matrices and $c \in L^\infty$ satisfying $c \geq 0$ or $\|c\|_{L^\infty} \ll 1$ (depending on the ellipticity of $A$). Let $u$ be a solution to $Lu = 0$ in $B_1(0)$. For every $\varepsilon > 0$, there exists $v \in W^{1,2}(B_1(0))$ such that $\|v - u\|_{L^2(\partial B_1(0))} < \varepsilon$, $Lv = 0$ in $B_1(0)$, $v(0) = u(0)$, and $\{v = 0\} \cap B_{1/2}(0)$ is a smooth manifold.
In fact, we are able to estimate the difference between $u$ and $v$ with additional control from below (see Theorem 5.4). The method that we develop can also be applied to more general domains than the unit ball and prove that \( \{v = 0\} \) is smooth inside of any relatively compact subset of the larger domain. We discuss these extensions and the assumptions on the coefficients of $L$ in Section 1.5. Let us first outline our proof of Theorem 1.1 in the case where $L = -\Delta$, the Laplacian.

1.1. **Strategy of the Argument.** Let $-\Delta u = 0$ and recall that \( \{u = 0\} \) is smooth wherever $\nabla u(x) \neq 0$. It is a classical result that critical points of analytic functions are isolated in level sets, i.e. if $\nabla u(x_i) = 0$ and $x_i \to x_0$, then the sequence $u(x_i)$ is eventually constant. Thus, \( \{u + \varepsilon = 0\} \cap B_{1/2}(0) \) is smooth for almost every $\varepsilon \in \mathbb{R}$. However, the condition that $v(0) = u(0)$ prevents us from simply taking $v \equiv u + \varepsilon$.

The difficulty of non-constant perturbations is that they may have singular points that are arbitrarily close to the original singular points. For example, in $\mathbb{R}^2$, while the nodal set of $u(x, y) = x^2 - y^2$ is smooth except at the origin, the nodal set of the perturbation

\[
v \equiv u + \varepsilon(x + y) = x^2 - y^2 + \varepsilon x + \varepsilon y \quad \text{(for any } \varepsilon \neq 0)\]

is smooth at the origin, but contains a singularity at $(x, y) = (-\varepsilon/2, \varepsilon/2)$.

Our strategy is to control where new singularities can arise when we perturb by a linear function $\varepsilon(e \cdot x)$ with $e \in S^{n-1}$ and $x \in \mathbb{R}^n$. Let $v \equiv u + \varepsilon(e \cdot x)$ and suppose that $v(x_0) = 0 = |\nabla v(x_0)|$, so that

\[
(1.1) \quad u(x_0) = -\varepsilon x_0 \cdot e \quad \text{and} \quad \nabla u(x_0) = -\varepsilon e.
\]

If we can prove that for all $x_0$ in a neighborhood of \( \{u = 0\} \),

\[
(1.2) \quad |u(x_0)| \leq c|\nabla u(x_0)|^\gamma \quad \text{for some } \gamma > 1,
\]

then the points for which (1.1) holds are a subset of \( \{x \in \mathbb{R}^n \mid |x \cdot e| \leq c\varepsilon^{\gamma-1}\} \). Repeating this argument with different directions $e \in S^{n-1}$ and increasingly smaller $\varepsilon > 0$ produces a perturbation with a smooth nodal set inside of $B_{1/2}(0) \setminus B_{\varepsilon}(0)$. A slightly different argument is needed to ensure \( \{v = 0\} \) is smooth near the origin.

The inequality (1.2) is known as a “Lojasiewicz gradient inequality” and holds for all analytic functions (see below). However, to prove our result in generality we need a version of (1.2) with control on the constants and for solutions to $-\text{div}(A\nabla u) + cu = 0$, which may not be analytic.

1.2. **Effective Lojasiewicz Inequalities.** The Lojasiewicz “distance” and “gradient” inequalities quantify the fact that analytic functions vanish to finite order and that the critical points of analytic functions are isolated in level sets:

**Theorem 1.2** (Lojasiewicz Inequalities [Loj59]). Let $f : \mathbb{R}^n \to \mathbb{R}$ be analytic. For any $x_0 \in \mathbb{R}^n$, there exists a neighborhood $U \ni x_0$, a constant $c > 0$, and an exponent $k \geq 1$ depending on $f$ such that

\[
(1.3) \quad |f(x) - f(x_0)| \geq c \text{dist}(x, \{y \mid f(y) = f(x_0)\})^k \quad \text{for all } x \in U.
\]
If \( \nabla f(x_0) = 0 \), then there also exists a constant \( \tilde{c} > 0 \) and exponent \( \theta \in (1, 2] \) such that
\[
|f(x) - f(x_0)| \leq \tilde{c} |\nabla f(x)|^\theta \quad \text{for all } x \in U.
\]

Originally developed to study the structure of analytic varieties, Łojasiewicz inequalities have found myriad applications in proving the uniqueness of tangents at singular points for geometric minimizers and the uniqueness of long time behavior for geometric flows (see e.g. [Sim83, CM15, Bre05]). While the inequalities in Theorem 1.2 are extremely powerful, they are also non-effective. That is to say, given a class of analytic functions \( \{f_\alpha\} \), it is very difficult and may not be possible to find a constant \( c > 0 \) and exponents \( k, \theta \) such that (1.3) and (1.4) hold with \( c, k, \) and \( \theta \) for all \( f_\alpha \). E.g., see [Kol99, Son12] for some partial results concerning families of polynomials. For additional background on the Łojasiewicz inequalities, interesting examples of (1.4) holding for polynomials with large exponents, and several perspectives on the proofs of (1.3) and (1.4), we recommend the paper of Feehan [Fee19].

In [BET17], the authors proved a version of (1.3) for harmonic polynomials of bounded degree with precise control on the Łojasiewicz constants. In this paper, we are able to extend our argument in [BET17] to apply to solutions of a general class of elliptic PDE with bounded supremum and bounded “frequency” at large scale (see Theorem 3.2 and the discussion around it for details). From this effective “distance inequality”, we employ a compactness argument to prove a quantitative Łojasiewicz “gradient inequality”:

**Theorem 1.3** (Effective Łojasiewicz Gradient Inequality). Set \( A \) to be an \( M \)-Lipschitz continuous \( n \times n \) matrix-valued function with \( \lambda I \leq A(x) \leq \Lambda I \) for all \( x \in B_2(0) \), where \( 0 < \lambda < \Lambda < \infty \). Let \( b \in L^\infty(B_2(0), \mathbb{R}^n) \), \( c \in L^\infty(B_2(0)) \) with \( \|b\|_{L^\infty}, \|c\|_{L^\infty} \leq M < \infty \) and define
\[
Lu \equiv -\text{div}(A \nabla u) + b \cdot \nabla u + cu.
\]
If \( u \) solves \( Lu = 0 \) in \( B_2(0) \) and if
\[
\frac{2}{\omega_n} \int_{\partial B_2(0)} |\nabla u|^2 u^2 \, d\sigma \leq N \leq N_0 \quad \text{and} \quad \int_{\partial B_1(0)} |u|^2 \, d\sigma = 1,
\]
then there exists a neighborhood, \( U \), of \( \{u = 0\} \) and constants \( c_1 = c_1(n, M, \lambda, \Lambda) > 0 \) and \( c_2 = c_2(n, M, \lambda, \Lambda, N_0) > 0 \) such that
\[
|\nabla u(y)| \geq \frac{c_1}{c_1 N_0} |u(y)| \quad \text{for all } y \in U \cap B_{1/2}(0).
\]

One might ask whether one can find a lower bound on the size of the neighborhood \( U \) in (1.6) that is uniform over the functions considered. However, even in the case of harmonic polynomials of degree at most 3, such a theorem is false.

**Example 1.4** (No Uniform Neighborhoods). Let \( h_\varepsilon : \mathbb{R}^3 \to \mathbb{R} \) be given by
\[
h_\varepsilon(x, y, z) = \varepsilon(x^2 - y^2) + x^3 - 3xy^2 + z^3 - 1.5(x^2 + y^2)z,
\]
where \( c_\varepsilon \) is chosen so that \( \int_{\partial B_1(0)} h_\varepsilon^2 \, d\sigma = 1 \). Note that \( c_\varepsilon \) is bounded above and away from zero uniformly whenever \( |\varepsilon| \lesssim 1 \). A straightforward calculation shows that \( |\nabla h_1(p)| = 0 \) and \( h_1(p) \neq 0 \), where \( p = \left( -\frac{4}{3} - \frac{2}{3}\sqrt{2}, 0, -\frac{2}{3} - \frac{2}{3}\sqrt{2} \right) \). Then, by a scaling argument,

\[
|\nabla h_\varepsilon(\varepsilon p)| = c_\varepsilon c_1^{-1} \varepsilon^2 |\nabla h_1(p)| = 0 \quad \text{and} \quad h_\varepsilon(\varepsilon p) = c_\varepsilon c_1^{-1} \varepsilon^3 h_1(p) \neq 0.
\]

From here it is clear that no inequality of the type (1.6) can hold in a fixed neighborhood of 0 for every \( h_\varepsilon \) with \( |\varepsilon| \ll 1 \).

Example 1.4 suggests that the gradient inequality is more delicate than the distance inequality (Theorem 3.2). Indeed, even in the case of an isolated critical point, where the gradient inequality, (1.4), can be derived from the distance inequality, (1.3), the proof of that implication requires applying the distance inequality to the directional derivatives of the original function (see [CM15]). Directional derivatives are analytic whenever the original function is analytic. However, it is not the case that the directional derivatives of \( u \) satisfy an equation of type (1.5) whenever \( u \) does. As such, even in the case of isolated critical points, we need to argue differently to establish a gradient inequality.

We believe Theorems 3.2 and 1.3 are the first known “effective” Lojasiewicz inequalities for a class of functions which is broader than polynomials. In fact, we note that solutions of \( Lu = 0 \) with \( L \) as in (1.5) may not be analytic (or even \( C^2 \)). In general, Lojasiewicz inequalities do not hold for functions which are merely in \( C^\infty \). In this vein, we point out related work of Colding and Minnicozzi in [CM19], where it was shown that \( C^2 \) solutions to a certain degenerate elliptic PDE also satisfy the Lojasiewicz inequalities, though their method of proof is substantially different from ours. It would be interesting to investigate the relationship between these approaches and, in the vein of [CM19], to see if Theorems 3.2 and 1.3 have any novel consequences for the behavior of gradient flows.

Key to our proofs of Theorems 3.2 and 1.3 is the Almgren frequency formula. Morally, the Almgren frequency formula and the Lojasiewicz inequalities both quantify the fact that the functions in question cannot vanish to arbitrarily high order, though we have not seen them connected anywhere else in the literature. Indeed, the “usual” proofs of (finite-dimensional) Lojasiewicz inequalities require tools from algebraic geometry like the resolution of singularities (cf. [Fee19]). However, the Almgren frequency formula is commonly used to understand the size and structure of nodal sets to elliptic PDEs.

1.3. Prior Results for (Generic) Nodal Sets. The literature concerning the size and structure of nodal sets to elliptic PDE is too vast to give it justice here, so we focus on the monotonicity formula/doubling index approach. For a comprehensive discussion, see the exposition of Han and Lin [HL13] or the survey of Logunov and Malinnikova [LM19].

The implicit function theorem implies that to understand the size and structure of \( \{ u = 0 \} \), it helps to understand the size and structure of the singular set,

\[
\mathcal{S}(u) := \{ x \mid u(x) = 0 \text{ and } |\nabla u(x)| = 0 \}.
\]

\footnote{In the final preparation of this manuscript, we learned that Lin and Lin [LL21] independently give a proof of the Theorem 3.2 (distance inequality) in the case that \( b = c = 0 \).}
It was first observed by Hardt and Simon [HS89] (later refined by Hoffmann-Ostenhof et al. [HOHON95, HOHON96, HHOHON99]) that a bound on the “doubling index”, \( \frac{\int_{\partial B_{2r}(x_0)} u^2}{\int_{\partial B_{r}(x_0)} u^2} \), of a solution or its gradient yields size estimates on the nodal or singular set. A key observation of Garofalo and Lin [GL86] is that the doubling index on small scales is controlled by the Almgren frequency formula at a single larger scale (see Theorem 2.5). Therefore, the size of the nodal or singular sets can be bounded by a function of the value of the frequency formula at some larger scale. This approach has been subsequently refined and adapted extensively to study related questions, including recently the incorporation of quantitative stratification ideas in [CNV15], the successful resolution of Yau’s conjecture in [Log18], and the study of boundary unique continuation in \( C^1 \) domains in [Tol21].

While we are not aware of any previous work on the regularity of generic solutions to the Dirichlet problem for elliptic PDE, there has been much interest in the generic behavior of eigenfunctions of the Laplace-Beltrami operator in domains or on manifolds. For example, using infinite dimensional Morse-Smale theory, Uhlenbeck [Uhl76] proved that for generic (in the Baire category sense) \( C^\infty \) metrics and eigenvalues, the associated eigenfunctions of the Laplace-Beltrami operator have smooth nodal sets. One could try to adapt this approach to the Dirichlet problem, however, a naive attempt to do so will fail as the dependence of the solution on the Dirichlet data does not enjoy the same functional-analytic properties as the dependence of the solution on the operator.

There has also been a large interest in studying “random” nodal sets of eigenfunctions to Laplace-Beltrami operators on specific (symmetric) manifolds and their connections to geometry and arithmetic, see e.g. [Jun20, CS19]. Particularly related to the present work is [NS09], in which Nazarov and Sodin prove that the nodal curves of a random spherical harmonic are smooth. (Further, they obtain estimates on the number of nodal domains of a random harmonic with sharp asymptotics.) We imagine the ideas of [NS09] could be adapted to prove a version of our Theorem 1.1 for constant coefficient operators. However, the connection between eigenvalues of the Laplace-Beltrami operator on \( \partial \Omega \) and solutions to the Dirichlet problem in \( \Omega \) is not so clear when the operator has variable coefficients or the domain, \( \Omega \), lacks symmetry. Thus, Theorem 1.1 does not seem to be related to the results in [NS09] for more general domains or equations.

1.4. Contrasts with “One-sided” Perturbations. Although there has not been much work on the “generic” Dirichlet problem for solutions to elliptic PDE, the question of whether minimizers of a functional are smooth for generic Dirichlet data has been studied in a variety of other contexts; e.g., see the recent breakthroughs [FRROS20, FRRO21] on obstacle-type problems and classical work [AL88, HS85] on harmonic maps and minimal hypersurfaces. Many of these results work with “one-sided” perturbations, where the Dirichlet data of the perturbation is strictly greater than (or lies strictly to one side of) the Dirichlet data of the original minimizer. See also related work for parabolic problems when the flow is “monotone” in some sense (e.g. [CCMS20, CM16, FROS21]).
An advantage of using one-sided perturbations is that the maximum principle and Harnack’s inequality often implies that the minimizer for this perturbed data also lies on one side of the original minimizer. One can then use geometric arguments to show that the perturbed minimizer is smooth (see e.g. [FROS20, HS85]). Such arguments are not sensitive to the precise nature of the perturbation and it is often the case that every one-sided perturbation results in a smooth minimizer. It may even suffice that the perturbation be “mostly” one-sided; e.g. [McI87] shows that every perturbation of an area-minimizing cone with isolated singularity whose infinitesimal generator has large projection onto a constant results in a smooth area-minimizing hypersurface. Constructing “mean zero” or “unsigned” de-singularizing perturbations is significantly more delicate, a naive reason being that one must avoid rotations of the original singular minimizer. Furthermore, it was shown in [CHS84] that there exist mean zero perturbations of area-minimizing cones with isolated singularity that remain singular and are not rotations of the original cone. (The perturbations are not even cones!)

In the context of the Dirichlet problem for a purely second order elliptic PDE on a ball, “mean zero” is captured by requiring that the perturbed function \( v \) has the same value as the original function \( u \) at the center the of the ball. This rules out simply adding a constant, which as mentioned above, is known to result in a smooth nodal set when \( L \) has real analytic coefficients. However, we should point out that when the coefficients of \( L \) are not analytic, we believe Theorem 1.1 is new even without the “mean zero” constraint. Again, we remark that it is possible to construct mean zero perturbations of homogeneous harmonic polynomials that have non-smooth nodal sets (see Example 1.4). Thus, we cannot hope to prove regularity for arbitrary mean zero perturbations.

Despite the extra care that is required, “mean zero” perturbations are a natural class to consider in many geometric contexts, for other types of boundary data (such as Neumann), for vector valued problems, and for parabolic flows in which \( \partial_t u \) does not have a sign. We hope to apply the methods established in this paper in other settings in the future.

1.5. Assumptions and Extensions. Let us now quickly discuss the assumptions and some possible extensions of our main theorems. The most crucial of these assumptions is the Lipschitz regularity of the matrix valued coefficient \( A \). There are examples of solutions to uniformly elliptic PDE with Hölder continuous coefficients (see e.g. [Pli63, Mil73]), which do not satisfy the strong unique continuation property. Because the Lojasiewicz inequalities imply the strong unique continuation property, it must therefore be the case that Lipschitz regularity on \( A \) or close cousin is needed to prove Theorems 1.3 and 3.2.

We did not investigate whether we might weaken the \( L^\infty \) conditions on \( b, c \) in Theorems 1.3 and 3.2 but it is not hard to imagine that it would suffice to have \( b \in L^p \) and \( c \in L^{2p} \) for some \( p > n \). Finally, it might be interesting to understand if effective Lojasiewicz inequalities could be proven for degenerate elliptic PDEs whose coefficients are subject to some scale invariant growth conditions, say that of Fabes-Kenig-Serapioni [FKS82].

In terms of Theorem 1.1, we restrict to the case where \( A \) is symmetric, \( b \equiv 0 \), and \( c \geq 0 \) or \( \|c\|_{L^\infty} \ll 1 \) in order to invoke the “ratio trick” (Lemma A.1) and reduce to the purely
second order case. We believe, though we did not investigate, that similar results could hold in the presence of non-zero $b \in L^\infty$, as long as the Dirichlet problem for both $L$ and its adjoint operator $L^*$ are uniquely solvable for all data in the relevant domains.

A minor modification of our arguments (see Remark 5.3) shows that in Theorem 1.1 we can take $B_2(0)$ to be any convex or Lipschitz domain and demand that the perturbed nodal set to be smooth in any compact subset of said domain. An interesting question is whether the nodal set of the perturbed solution can be made smooth all the way to the boundary. This is related to recent work on boundary unique continuation [Tol21, KZ21].

Finally, one might ask if the “critical set”, \( \{ \nabla u = 0 \} \), can be (locally) perturbed away. However, this is not possible in general, as we now explain.

In the plane, any function on the unit circle that changes sign four or more times will have a harmonic extension with non-empty critical set. (It must have an interior saddle point.) By continuity, any critical points of a perturbed function must be close to critical points of the original function. Also, the property of changing sign four or more times is stable under perturbation. Putting all of the pieces together, we conclude that any perturbation of a harmonic function with Dirichlet data on $S^1$ that changes sign four or more times and has a unique critical point at the origin must also have a critical point close to the origin. In particular, it is impossible to perturb away the critical set of $u \equiv x^2 - y^2$.

**Acknowledgements:** M.E. thanks David Jerison, Carlos Kenig and Vladimir Šverák for encouraging conversations at the beginning of this project. He also thanks David Jerison for making him aware of [Uhl76]. The authors thank Stefan Steinerberger and Alexander Logunov for their helpful comments. Logunov suggested the “ratio trick” (Lemma A.1), which has allowed us to simplify some arguments from an earlier draft of the manuscript and strengthen the main theorem.

2. Preliminaries and Doubling

In this section, we gather some preliminary results on the solutions to second-order divergence form elliptic PDEs with Lipschitz leading coefficients. These are well known to experts, but to help readers from adjacent fields we include the relevant results here. Let $A : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ be a matrix-valued function. We assume that $A(x)$ is symmetric for each $x \in \mathbb{R}^n$, and with respect to the Euclidean norms and inner product,

\begin{equation}
|A(x) - A(y)| \leq M|x - y|, \quad \lambda |\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda |\xi|^2,
\end{equation}

for all $x,y,\xi \in \mathbb{R}^n$, where $0 \leq M < \infty$ and $0 < \lambda \leq \Lambda < \infty$. For the remainder of the paper, we reserve $M, \lambda, \Lambda$ to denote the constants associated to the Lipschitz regularity and ellipticity of $A$. We are interested in general divergence form second order elliptic PDE with Lipschitz second order term $A$ and bounded lower order terms $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}$:

\begin{equation}
Lu = -\text{div}(A\nabla u) + b \cdot \nabla u + cu,
\end{equation}

with $A$ as in (2.1) and $b, c \in L^\infty$.

\footnote{The authors thank Stefan Steinerberger for suggesting this argument.}
Theorem 2.2. We write $L \in \mathcal{L}(M, \lambda, \Lambda)$ if $L$ is as in (2.2) and if the lower order terms satisfy $|b(x)| \leq M$ and $|c(x)| \leq M$ for all $x \in \mathbb{R}^n$. We say that $u$ is a weak solution to $Lu = 0$ (or $u$ is a weak solution of $L$) in an open set $\Omega \subset \mathbb{R}^n$ if $u \in W^{1,2}(\Omega)$ and

$$\int_{\Omega} A \nabla u \cdot \nabla \varphi + b \cdot \varphi \nabla u + cu \varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

In the sequel, all solutions to $Lu = 0$ are assumed by default to be weak solutions. Let us recall the additional regularity enjoyed by weak solutions of $Lu = 0$ when $L \in \mathcal{L}(M, \lambda, \Lambda)$. We may write $\int_A f \, d\mu \equiv \mu(A)^{-1} \int f \, d\mu$ if convenient.

**Theorem 2.2** (see [HL11] Theorem 3.13). Let $L \in \mathcal{L}(M, \lambda, \Lambda)$. If $Lu = 0$ in $B_r(0)$ for some $0 < r \leq 2$, then for every $\alpha \in (0, 1)$,

$$\|u\|_{C^{1,\alpha}(B_{r/2}(0))} \leq C \left( \frac{1}{r} \left( \int_{B_r(0)} |u|^2 \right)^{1/2} + \left( \int_{B_r(0)} |\nabla u|^2 \right)^{1/2} \right)$$

for some constant $C$ depending only on $n$, $M$, $\lambda$, $\Lambda$, and $\alpha$.

**Proof Sketch.** When $b = 0$ this is [HL11] Theorem 3.13 (written in a scale invariant manner). We sketch the minor modifications needed when $b \in L^\infty$.

We set $r = 1$ and let $B\rho(x_0) \subset B_1(0)$ and let $w$ solve

$$\kern-10pt -\text{div}(A(x_0)\nabla w) = 0 \text{ in } B\rho(x_0) \quad \text{and} \quad w = u \text{ on } \partial B\rho(x_0).$$

Then $v = u - w \in W^{1,2}_0(B\rho(x_0))$ solves

$$\int_{B\rho(x_0)} a_{ij}(x_0) \nabla v \nabla \varphi \, dx = \int_{B\rho(x_0)} (a_{ij}(x_0) - a_{ij}) \nabla u \nabla \varphi - (b \nabla u + cu) \varphi \, dx$$

for all $\varphi \in W^{1,2}_0(B\rho(x_0))$.

Letting $\varphi = v$ in (2.4), using $L \in \mathcal{L}(M, \Lambda, \lambda)$ and Sobolev embedding (see a similar argument in [HL11]), we find that

$$\int_{B\rho(x_0)} |\nabla v|^2 \, dx \leq C\Big(\rho \|\nabla v\|_{L^2(B\rho(x_0))} \|\nabla u\|_{L^2(B\rho(x_0))} + \|v\|_{L^2(B\rho(x_0))} \|b\|_{L^\infty(B\rho(x_0))} \|\nabla u\|_{L^2(B\rho(x_0))} \Big) + C\|\nabla v\|_{L^2(B\rho(x_0))} \|c\|_{L^n(B\rho(x_0))} \|u\|_{L^2(B\rho(x_0))},$$

where $C = C(\Lambda, \lambda, M) > 0$.

Apply the Poincare inequality to $v$ in $B\rho(x_0)$, divide (2.5) by $\|\nabla v\|_{L^2(B\rho(x_0))}$ and square both sides to get

$$\int_{B\rho(x_0)} |\nabla v|^2 \, dx \leq C \left( \rho^2 (1 + \|b\|^2_{L^\infty(B\rho(x_0))}) \|\nabla u\|^2_{L^2(B\rho(x_0))} + \|c\|^2_{L^n(B\rho(x_0))} \|u\|^2_{L^2(B\rho(x_0))} \right);$$

where again $C = C(\Lambda, \lambda, M) > 0$. From (2.6) we argue as in [HL11] Theorem 3.13 to finish (where (2.6) plays the role of the equation at the top of page 61 in [HL11]).
A nice consequence of Theorem 2.2 and standard interior estimates for weak solutions (see [GT01, Theorem 8.24]) is compactness, facilitating normal families arguments:

**Corollary 2.3.** Suppose that $u_1, u_2, \ldots$ are solutions in $B_2(0)$ for a sequence of operators $L_1, L_2, \cdots \in \mathcal{L}(M, \lambda, \Lambda)$. If the $u_i$ are uniformly bounded in $W^{1,2}(B_2(0))$ or $L^\infty(B_2(0))$, and if $0 < \rho < 2$ and $\alpha \in (0, 1)$, then there exists an operator $L_0 \in \mathcal{L}(M, \lambda, \Lambda)$, a function $u_0 \in C^{1,\alpha}(B_\rho(0))$, and a subsequence of $(L_i, u_i)$ (which we relabel) such that

- $u_i \to u_0$ in $C^{1,\alpha}(B_\rho(0))$,
- $u_0$ is a weak solution to $L_0 u = 0$ in $B_\rho(0)$, and
- $A_i \to A_0$ uniformly in $B_2(0)$, and $b_i \rightharpoonup b_0$ and $c_i \rightharpoonup c_0$ in the weak-star topology on $L^\infty(B_2(0))$, where $A_i, b_i, c_i$ denote the second and lower order parts of $L_i$.

Whereas versions of Theorem 2.2 and Corollary 2.3 exist for more general operators than those in (2.2), the Lipschitz regularity of the leading coefficient is necessary for the strong unique continuation principle.

**Theorem 2.4 (Strong Unique Continuation Principle).** Let $L$ be as in (2.2). If $Lu = 0$ in $B_2(0)$ and $u \not\equiv 0$, then there exists $d \in \mathbb{N}$ (depending on $u$) such that

$$\liminf_{r \downarrow 0} \frac{\sup_{\partial B_r(x)} |u|}{r^d} = \infty$$

for all $x \in B_2(0)$. If $Lu = 0$ vanishes on some nonempty open subset of $B_2(0)$, then $u \equiv 0$ in $B_2(0)$.

As mentioned above, Theorem 2.4 does not hold under the assumption that $A$ is merely Hölder continuous; see [Pli63, Mil73]. Theorem 2.4 is actually a consequence of the (almost) monotonicity of Almgren frequency, which we discuss now.

**2.1. Almgren Frequency Function.** Given a real-valued function $u$ on $\mathbb{R}^n$ or an open subset, we define the associated \textit{Almgren frequency} $N$, first introduced in [Alm79]:

$$N(u, x_0, r) \equiv \frac{r \int_{B(x_0, r)} |\nabla u|^2 \, dx}{\int_{B(x_0, r)} u^2 \, d\sigma} \quad \text{for all } x_0 \in \mathbb{R}^n \text{ and } r > 0. \tag{2.7}$$

Almgren proved that if $u$ is harmonic in $B(x_0, 2r_0)$, then the function $r \mapsto N(u, x_0, r)$ is decreasing as $r \downarrow 0$ in $(0, r_0)$. There are also analogous quantities for solutions of the equations in (2.2) (see e.g. [GL87]). The key observation for us (and in many related problems) is that the frequency controls the “doubling ratio” of a harmonic function $u$. In turn, this is a quantification of the rate at which $u$ vanishes at any point. For a more in depth discussion of this phenomena, see [HL13, Chapter 3].

Below, we need similar control for our solutions to the elliptic operators given in (2.2). Fortunately, such control for variable coefficient operators which Lipschitz continuous first order terms is by now classical; see [GL87, Theorem 1.2]. The following theorem is a very slight modification of [HL13, Theorem 3.2.10], which we state and sketch the proof for the sake of completeness.
Theorem 2.5. Let \( u \) be a solution to \( Lu = 0 \) in \( B_2(x_0) \) for some \( L \in \mathcal{L}(M, \lambda, \Lambda) \). Then there exist a \( r_0 \in (0, 1), c_3, c_4 > 0 \) (which depend only on \( M, \lambda, \Lambda \) and the dimension \( n \)) such that for all \( p \in B_{3/4}(x_0) \) and all \( r < r_0 \) we have

\[
(2.8) \quad \frac{1}{B_{2r}(p)} \int_{B_{2r}(p)} u^2 \leq 2^{c_3 N(u, x_0, 2) + c_4} \frac{1}{B_r(p)} \int_{B_r(p)} u^2.
\]

Proof. Let \( u \) be a solution in \( B_2(x_0) \) as stated. From [HL13 (3.2.40)], simplifying and adapting to our situation, we know that there exists \( r_0 < 1/4 \) and \( k_1, k_2 \) depending on \( n, M, \lambda, \) and \( \Lambda \) such that if \( u \) satisfies \( Lu = 0 \) in \( B_1(p) \), then

\[
(2.9) \quad \frac{1}{B_{2r}(p)} \int_{B_{2r}(p)} u^2 \leq 2^{k_1 N(u, p, 4r_0) + k_2} \frac{1}{B_r(p)} \int_{B_r(p)} u^2,
\]

for all \( r < r_0 \). Since for all \( p \in B_{3/4}(x_0) \) we have \( B_1(p) \subset B_2(x_0) \), equation (2.8) will hold if we can show that there exists \( C > 0 \) (possibly depending on \( n, M, \lambda, \) and \( \Lambda \)) such that

\[
(2.10) \quad \sup_{p \in B_{3/4}(x_0)} N(u, p, 4r_0) \leq CN(u, x_0, 2) + 1.
\]

Assume that (2.10) fails. Then, after performing a harmless translation and scalar multiplication, we can find an operator \( L_i \in \mathcal{L}(M, \lambda, \Lambda) \) and a solution to \( L_i u_i = 0 \) in \( B_2(0) \) with \( \sup_{B_{2}(0)} |u_i| = 1 \), but \( N(u_i, p_i, 4r_0) \geq i N(u_i, 0, 2) + 1 \) for some \( p_i \in B_{3/4}(0) \). By Corollary 2.3, after passing to a subsequence, we can assume that \( p_i \to p_0 \in \overline{B}_{3/4}(0) \) and \( u_i \to u_0 \) \( C^{1, \alpha}(B_1(0)) \) for fixed \( \alpha \in (0, 1) \), where \( L_0 \in \mathcal{L}(M, \Lambda, \lambda) \) and \( L_0 u_0 = 0 \). There are now two cases, each of which leads to a contradiction.

Case 1. If \( N(u_i, 0, 2) \to 0 \), then \( u_0 \) is constant in \( B_1(0) \). Given the assumption \( \sup_{B_2(0)} |u_i| = 1 \), this implies that \( u_0 \neq 0 \). Thus, \( N(u_i, p_i, 4r_0) \to N(u_0, p_0, 4r_0) = 0 \), which contradicts a contradiction as \( N(u_i, p_i, 4r_0) \geq N(u_i, 0, 2) + 1 \geq 1 \).

Case 2. Otherwise, if \( \limsup N(u_i, 0, 2) > 0 \), then we discover that \( N(u_0, p_0, 4r_0) = \infty \). It follows that \( u_0 \equiv 0 \) on \( \partial B_{4r_0}(p_0) \), which contradicts [HL13 Corollary 3.2.5] when \( r_0 \) is sufficiently small. \( \square \)

With Theorem 2.5 in hand, one can prove the following blowup version of Corollary 2.3. The control on the relevant norms is slightly weaker.

Corollary 2.6. Let \( u_1, u_2, \ldots \) be solutions in \( B_2(0) \) for operators \( L_1, L_2, \ldots \in \mathcal{L}(M, \lambda, \Lambda) \). Assume that \( N(u_i, 0, 2) \leq N_0 < \infty \) for all \( i \). Given \( x_i \in \{u_i = 0\} \cap B_{3/4}(0) \) and \( r_i \downarrow 0 \), define

\[
\tilde{u}_i(z) = \frac{u_i(r_iz + x_i)}{\left( \int_{B_{2r_i}(x_i)} |u_i|^2 \right)^{1/2}} \quad \text{or} \quad \tilde{u}_i(z) = \frac{u_i(r_iz + x_i)}{\left( \int_{\partial B_{2r_i}(x_i)} |u_i|^2 \right)^{1/2}}.
\]

Then there exists \( L_\infty \in \mathcal{L}(M, \lambda, \Lambda) \) of the form \( Lu \equiv -\operatorname{div}(A_\infty \nabla u) \), i.e. \( L_\infty \) is without lower order terms, and a solution to \( L_\infty u_\infty = 0 \) such that for every \( \alpha \in (0, 1) \) and \( R > 0 \), we have \( \tilde{u}_i \to u_\infty \) in \( C^{1, \alpha}(B_R(0)) \) along some subsequence of \( \tilde{u} \).

Related to Corollary 2.6 is a “reverse Hölder inequality”: 
**Corollary 2.7.** Let $u$ be a solution to $Lu = 0$ in $B_2(0)$ for $L \in \mathcal{L}(M, \Lambda, \lambda)$ with $N(u, 0, 2) \leq N_0$ for some $N_0 > 0$. Fix $\rho \in (0, 1/4)$. Then there exists a constant $K = K(N_0, M, \Lambda, \lambda, \rho) > 0$ such that for all $\tilde{p} \in B_{3/4}(0)$ we have

$$\int_{B_{r}(\tilde{p})} |u|^2 \, dx \geq K \sup_{B_1(0)} |u|^2.$$  

**Proof.** Assume not, and obtain a contradictory sequence of $u_i, p_i$ which satisfy $Lu_i = 0$ in $B_2(0)$ $p_i \in B_{3/4}(0)$ with $N(u_i, 0, 2) \leq N_0$ and $\sup_{B_2(0)} |u_i|^2 = 1$ (after harmless scalar multiplication) but

$$\int_{B_{r}(p_i)} |u_i(x)|^2 \leq \frac{1}{i} \sup_{B_1(0)} |u_i|^2.$$  

Invoking Theorem 2.2, Corollary 2.3 and passing to a subsequence we get that $p_i \to p_0$ and $u_i \to u_0$ in $C^{1,\alpha}(B_1(0))$, and weakly in $W^{1,2}(B_2(0))$ with $L_0 u_0 = 0$ in $B_2(0)$ for some $L_0 \in \mathcal{L}(M, \lambda, \lambda)$. This implies that $\sup_{B_1(0)} |u_0|^2 < \infty$, thus, by the offset equation above, $\int_{B_{r}(p_0)} |u_0|^2 = 0$. By the strong unique continuation principle, Theorem 2.4 applied to $L_0$, $u_0 \equiv 0$ in $B_2(0)$ which contradicts $\sup_{B_2(0)} |u_0|^2 \equiv 1$ (by the continuity of the trace). So the constant $K$ exists and we get the theorem. \[\square\]

### 3. Lojasiewicz Distance Inequalities

In this section, we establish Lojasiewicz distance inequalities for solutions of [2.2] with bounds that are uniform over the coefficients of the operator and large scale bounds on the size and frequency of the solution. These results generalize our prior work in [BET17], which we recall here:

**Theorem 3.1** (Distance Lojasiewicz for Harmonic Polynomials [BET17, Theorem 3.1]). For all $n \geq 2$ and $k \geq 1$, there exists a constant $c = c(n, k) > 0$ with the following property. If $p : \mathbb{R}^n \to \mathbb{R}$ is a non-constant harmonic polynomial of degree at most $k$ and $x_0 \in \{p = 0\}$, then

$$|p(z)| \geq c\|p\|_{L^\infty(B_1(x_0))} \text{dist}(z, \{p = 0\})^k \quad \text{for all } z \in B_{1/2}(x_0).$$

The exponent in (3.1) is optimal. Note that if $p$ is a harmonic polynomial and $\deg p \leq k$, then $N(p, x_0, r) \leq k$ for all $r > 0$ and $x_0 \in \mathbb{R}^n$. (In fact, harmonic functions with this property are necessarily polynomials.) The following generalization to a broader class of solutions of second-order operators highlights the importance of bounded frequency in the derivation of the distance Lojasiewicz inequality. The class $\mathcal{L}(M, \lambda, \Lambda)$ of valid operators is specified in Definition 2.1.

**Theorem 3.2** (Distance Lojasiewicz Inequality for Solutions). For all $n$, $M$, $\lambda$, $\Lambda$, and $N_0$, there exist constants $c_5, c_6, r_1 > 0$, with $c_5$ independent of $N_0$, so the following holds. If $L \in \mathcal{L}(M, \lambda, \Lambda)$ and $u$ is a solution to $Lu = 0$ in $B_2(x_0)$ with $N(u, x_0, 2) \leq N_0$, then for all $p \cap B_{1/2}(x_0)$ such that $u(p) = 0$, 

$$|u(z)| \geq c_6\|u\|_{L^\infty(B_1(x_0))} \text{dist}(z, \{u = 0\})^{c_5 N(u, x_0, 2)} \quad \text{for all } z \in B_{r_1}(p).$$
Proof. Let \( p \in B_{1/2}(x_0) \) with \( u(p) = 0 \) and \( z \in B_{r_0}(p) \) where \( r_0 > 0 \) is the radius in Theorem 2.5. Let \( \tilde{p} \) be such that \( u(\tilde{p}) = 0 \) and let \( r_1 \in (0, 1) \) be such that \( r_0/2 \geq r_1/2 > r \equiv |z - \tilde{p}| = \text{dist}(z, \{u = 0\}) \). We let \( r_1 \) be small enough (but independent of \( u, z, p, x_0 \)) to ensure that \( \tilde{p} \in B_{3/4}(x_0) \).

Note that \( u \) has a sign in \( B_{r_0/2}(z) \). To proceed, first apply the Harnack inequality once and then Theorem 2.5 \([\log_2(r_1/\rho)] \) times. Observing that \( 2^{-[\log_2(r_1/\rho)]} \geq \frac{1}{2}(\rho/r_1) \) and \( B_{r_1/2}(z) \subset B_{r_1}(\tilde{p}) \), we have

\[
|u(z)|^2 \geq c \int_{B_{r_0/2}(z)} |u(x)|^2 \geq c 2^{-[\log_2(r_1/\rho)](c_3 N(u,x_0,2) + c_4)} \int_{B_{r_1}(\tilde{p})} |u(x)|^2 \geq c \left( \frac{\rho}{r_1} \right)^{2(c_3 N(u,x_0,2) + c_4)} \int_{B_{r_1}(\tilde{p})} |u(x)|^2,
\]

where for the duration of the proof we let \( c > 0 \) denote a constant (not a term of \( L \)) that may change value from line to line, but always depends on at most \( n, M, \lambda, \Lambda \), and \( N_0 \), whereas \( c_3, c_4 > 0 \) denote the constants in Theorem 2.5. Since \( \rho \leq |z - p| \leq r_1 \), we can incorporate the term \((\rho/r_1)^{c_4}\) into \( c \) (by reducing the value of \( c \)). Also, because \( r_1 < 1 \), we can bound \( r_1^{-c_3 N(u,x_0,2)} \) from below by \( 1 \). Thus, recalling the definition of \( \rho \), we have

\[
|u(z)| \geq c \text{dist}(z, \{u = 0\})^{c_3 N(u,x_0,2)} \left( \int_{B_{r_1}(\tilde{p})} |u(x)|^2 \right)^{1/2}
\]

where \( c_5 = 2c_3 \) depends only on \( n, M, \lambda, \Lambda \) (and not on \( N_0 \)).

We then note that (3.2) follows from Corollary 2.7 letting \( K = K(M, \lambda, \Lambda, N_0, r_1) > 0 \) and setting \( c_6 = cK \).

\[\square\]

4. Lojasiewicz Gradient Inequalities

We are now ready to prove our main Lojasiewicz inequality via a compactness argument:

**Theorem (Theorem 1.3 Restated).** Let \( L \in \mathcal{L}(M, \lambda, \Lambda) \). If \( u \) solves \( Lu = 0 \) in \( B_2(0) \), \( N(u,0,2) \leq N_0 \), and \( \int_{\partial B_1(0)} |u|^2 \, d\sigma = 1 \), then there exists a neighborhood, \( U \), of \( \{u = 0\} \) and constants \( c_1 = c_1(n, M, \lambda, \Lambda) > 0 \) and \( c_2 = c_2(n, M, \lambda, \Lambda, N_0) > 0 \) such that

\[
(4.1) \quad |\nabla u(y)|^{\frac{c_1 N_0}{c_1 N_0 - 1}} \geq c_2 |u(y)| \quad \text{for all} \quad y \in U \cap B_{1/2}(0).
\]

**Proof.** Set \( c_1 = c_5 \), where \( c_5 \) is the constant in Theorem 3.2. Assume the theorem fails for this choice of \( c_1 \). Then there exist functions \( u_i \), which solve equations \( L_i u_i = 0 \) in \( B_2(0) \), that satisfy \( N(u_i,0,2) \leq N_0 \), \( \int_{\partial B_1(0)} u_i^2 \equiv 1 \), and such that for each \( u_i \) there is a sequence of points \( x_{i,j} \in B_{1/2}(0) \) with \( x_{i,j} \to \{u_i = 0\} \) as \( j \to \infty \), where

\[
(4.2) \quad |\nabla u_i(x_{i,j})|^{\frac{c_1 N_0}{c_1 N_0 - 1}} \leq \frac{1}{i} |u_i(x_{i,j})|.
\]

For each \( i \) and \( j \), let \( Q_{i,j} \) denote the closest point in \( \{u_i = 0\} \) to \( x_{i,j} \) (if there is more than one, choose arbitrarily) and let \( 2r_{i,j} = |x_{i,j} - Q_{i,j}| \). Note that for each \( i \), we have \( \lim_{j \to \infty} r_{i,j} = 0 \). Build a diagonal subsequence. In particular, for each \( i \), choose \( j_i \) such
that \( r_i := r_{i,i} \downarrow 0 \) and set \( x_i := x_{i,i} \) and \( Q_i := Q_{i,i} \). Our immediate goal is to show that an inequality close to (4.2) holds at every point \( y \in B_{r_i/2}(x_i) \).

Applying Theorem 2.2 for any choice of \( \alpha \in (0,1) \), there exists a \( C > 0 \) such that

\[
|\nabla u_i(y)| \leq |\nabla u_i(x_i)| + Cr_i^\alpha \left( \frac{1}{r_i} \left( \int_{B_{r_i}(x_i)} |u_i|^2 \right)^{1/2} + \left( \int_{B_{r_i}(x_i)} |\nabla u_i|^2 \right)^{1/2} \right)
\]

for all \( y \in B_{r_i/2}(x_i) \). Put \( \gamma := \frac{\alpha}{\gamma \alpha \alpha \alpha \alpha - 1} > 1 \) and apply (4.2), Jensen’s inequality, and Harnack’s inequality to get that for some \( C > 0 \) (depending on \( \gamma, \alpha, M, \Lambda, \lambda, n \)),

\[
|\nabla u_i(y)|^\gamma \leq \frac{C}{\gamma} |u_i(y)| + Cr_i^{\alpha \gamma} \left( \frac{1}{r_i} \left( \int_{B_{r_i}(x_i)} |u_i|^2 \right)^{1/2} + \left( \int_{B_{r_i}(x_i)} |\nabla u_i|^2 \right)^{1/2} \right)^\gamma
\]

for all \( y \in B_{r_i/2}(x_i) \). Divide both sides of (4.3) through by \( \left( \frac{1}{r_i} \left( \int_{B_{r_i}(x_i)} |u_i|^2 \right)^{1/2} \right)^\gamma \).

Then, observing that \( B_{r_i}(x_i) \subset B_{5r_i}(Q_i) \) and invoking interior \( W^{1,2} \) estimates,

\[
\left( \frac{r_i |\nabla u_i(y)|}{\left( \int_{B_{r_i}(x_i)} |u_i|^2 \right)^{1/2}} \right)^\gamma \leq \frac{C}{\gamma} \left( \int_{B_{r_i}(x_i)} |u_i|^2 \right)^{1/2} r_i^{\gamma \alpha \gamma} \left( \int_{B_{r_i}(x_i)} |\nabla u_i|^2 \right)^{(1-\gamma)/2} + Cr_i^{\alpha \gamma}
\]

for all \( y \in B_{r_i/2}(x_i) \). To proceed, define

\[
\tilde{u}_i(z) := \frac{u_i(r_i z + Q_i)}{\left( \int_{B_{r_i}(x_i)} |u_i|^2 \right)^{1/2}}.
\]

By Corollary 2.6 we can find a subsequence (which we relabel) along which \( \tilde{u}_i \to u_\infty \) in \( C^{1,\alpha}(B_2(0)) \), where \( u_\infty \) satisfies \(-\text{div}(A_0 \nabla u_\infty) = 0 \) in \( \mathbb{R}^n \) for constant elliptic matrix \( A_0 \).

Write \( \tilde{x}_i = \frac{x_i - Q_i}{r_i} \). By passing to a further subsequence, if necessary, we may assume that \( \tilde{x}_i \to x_\infty \) for some \( x_\infty \in B_2(0) \). Then (4.4) implies that

\[
|\nabla u_\infty(z)|^\gamma \leq \lim inf_{i \to \infty} \frac{C}{\gamma} r_i \left( \int_{B_{r_i}(x_i)} |u_i|^2 \right)^{(1-\gamma)/2}
\]

for all \( z \in B_{1/4}(x_\infty) \).

Next, suppose that we know

\[
\lim sup_i r_i^{\gamma \alpha \gamma} \left( \int_{B_{r_i}(x_i)} |u_i|^2 \right)^{(1-\gamma)/2} < \infty.
\]

If this is the case, then (4.5) implies that \( |\nabla u_\infty(z)| \equiv 0 \) in \( B_{1/4}(x_\infty) \). By Theorem 2.4 (the strong unique continuation principle), \( u_\infty \) is constant on \( \mathbb{R}^n \). On one hand, \( \tilde{u}_i(0) = 0 \) for all \( i \), so it must be the case that \( u_\infty \equiv 0 \). On the other hand, \( \text{dist}(\tilde{x}_i, \{ \tilde{u}_i = 0 \}) = 2 \), so by the Lojasiewicz by the distance inequality (Theorem 3.2), there exists a \( \kappa > 0 \) such that \( |\tilde{u}_i(\tilde{x}_i)| \geq \kappa \Rightarrow |u_\infty(x_\infty)| \geq \kappa/2 \) for all \( i \). This gives a contradiction (to the assumption at the beginning of the proof).
Thus, we have reduced the proof of the gradient inequality to showing that \((4.6)\) holds. Using the distance inequality again (and recalling that \(1 - \gamma < 0\)), we get that
\[
\left( \int_{B_{r_0}(Q_i)} |u_i|^2 \right)^{(1-\gamma)/2} \leq Cr_i^{c_5N_0(1-\gamma)},
\]
where \(c_5 > 0\) is the constant from Theorem 3.2. Thus, \((4.6)\) holds as long as
\[
\gamma + c_5N_0(1 - \gamma) \geq 0 \iff c_5N_0 \geq (c_5N_0 - 1)\gamma.
\]
This is true, because we chose \(c_1 = c_5\). Therefore, \((4.6)\) and Theorem 1.3 are true too. \(\square\)

Remark 4.1. The same argument using Theorem 3.1 instead of Theorem 3.2 yields a gradient Lojasiewicz inequality for harmonic polynomials in \(\mathbb{R}^n\) of degree at most \(k\) with the sharp exponent \(k/(k-1)\) instead of \(c_1k/(c_1k-1)\).

5. Generic Smoothness for Nodal Sets

We now use the Lojasiewicz gradient inequalities to prove that nodal sets for solutions to elliptic PDE are smooth for “most” Dirichlet data. Recall that we sketched this argument for harmonic functions in Section 1.1. The key idea is to perturb using linear functions and then use Theorem 1.3 to control possible locations that new singular points may arise.

To replicate these arguments for more general operators, we need a good analogue of the coordinate functions \(\{x_1, x_2, \ldots, x_n\}\), which solve \(Lu = 0\). For purely second order elliptic PDE in divergence form, such solutions are usually called “harmonic coordinates” and it is by now classical that these coordinates exist locally and have the desired size estimates (e.g. see [Pet16, Lemma 11.2.5]). Even more, it is known that locally defined harmonic coordinates can be approximated by globally defined solutions to \(Lu = 0\) using Runge approximation (e.g. see [GW73]). Unfortunately, we cannot take these results off the shelf as we need some quantitative control on our globally defined solutions outside of the neighborhood in which they approximate the coordinate functions. To achieve this, we use the quantitative Runge approximation results of [RS19].

Let us draw special attention to the fact that we no longer work in the full generality of \(L \in \mathcal{L}(M, \lambda, \Lambda)\). Our construction of harmonic coordinates will be for operators without drift terms and \(c \geq 0\). For the remainder of the paper, we define
\[
\mathcal{L}_{0,0}(M, \lambda, \Lambda) \equiv \{L \in \mathcal{L}(M, \lambda, \Lambda) : b = 0, c = 0\},
\]
\[
\mathcal{L}_{0,+}(M, \lambda, \Lambda) \equiv \{L \in \mathcal{L}(M, \lambda, \Lambda) : b = 0, c \geq 0\},
\]
and
\[
\mathcal{L}_{0,\delta}(M, \lambda, \Lambda) \equiv \{L \in \mathcal{L}(M, \lambda, \Lambda) : b = 0, \|c\|_{L^\infty} \leq \delta\},
\]
where \(\mathcal{L}(M, \lambda, \Lambda)\) is the class of operators defined in Definition 2.1.

Theorem 5.1 (Harmonic Coordinates). Let \(L \in \mathcal{L}_{0,+}(M, \lambda, \Lambda)\). There exist constants \(0 < r_0 \leq 1/4\) and \(C > 0\) depending only on \(n, M, \lambda, \) and \(\Lambda\) with the following property.
For every $p \in B_{1/2}(0)$ and $0 < r \leq r_{0}$, there exist functions $h_{1}, \ldots, h_{n} \in W^{1,2}(B_{1}(0))$ satisfying $Lh_{i} = 0$ in $B_{1}(0)$ and

$$\sup_{x \in B_{r}(p)} \|\nabla h_{i} - e_{i}\| < \frac{1}{n^{100}},$$

$$\int_{B_{1}(0)} |\nabla h_{i}|^{2} \leq C,$$

and also $\sup_{\partial B_{1}(0)} |h_{i}| \leq C$, and $Cr^{-1} \leq \|h_{i}\|_{L^{2}(\partial B_{1}(0))}$ for all $1 \leq i \leq n$. If $L$ is of the form $Lu \equiv -\text{div}(A\nabla u)$, then we can also ask that each $h_{i}(0) = 0$.

**Proof.** Let $Lu \equiv -\text{div}(A\nabla u) + cu \in L_{0,+}(M, \lambda, \Lambda)$ and $p \in B_{1/2}(0)$ be given. We claim that for any $\varepsilon_{0} > 0$, there is $0 < \rho_{0} \leq 1/4$ small enough such that if $0 < r < \rho_{0}$ and $\tilde{h}_{i}$ is the solution to the following Dirichlet problem,

$$L \tilde{h}_{i} = 0 \text{ in } B_{2r}(p) \text{ and } \tilde{h}_{i} = x_{i} \text{ on } \partial B_{2r}(p),$$

then $\|\nabla \tilde{h}_{i} - e_{i}\|_{L^{\infty}(B_{r}(p))} < \varepsilon_{0}$. To prove the claim (compare to [Pet16, Lemma 11.2.5] for the classical case of purely second-order operators), consider the associated blowup $u(x) = r^{-1} \left( \tilde{h}_{i}(rx + p) - (rx + p) \cdot e_{i} \right)$ and notice that for $x \in B_{2}(0)$,

$$-\text{div}(A(rx + p)\nabla u(x)) + r^{2}c(rx + p)u(x) = \text{div}((A(rx + p) - A(p))e_{i}) - rc(rx + p)((rx + p) \cdot e_{i}).$$

Standard estimates for the solutions to divergence form elliptic PDE and the maximum principal (e.g. see [GT01, Theorems 8.16 and 8.33]) tell us that $\|\nabla u\|_{L^{\infty}(B_{1}(p))}^{2}$ is bounded above by

$$C \left( \|(A(rx + p) - A_{0})e_{i}\|_{C^{1/2}(B_{0}(p))}^{2} + \|rc(rx + p)((rx + p) \cdot e_{i})\|_{L^{\infty}(B_{2}(0))}^{2} \right) \leq Cr,$$

where we have used the Lipschitz continuity of $A$ and $C > 0$ depends only $n$, $M$, $\lambda$, and $\Lambda$. Thus, taking $\rho_{0}$ small enough depending on $C$ and $\varepsilon_{0}$, (5.5) gives $\|\nabla \tilde{h}_{i} - e_{i}\|_{L^{\infty}(B_{r}(p))} < \varepsilon_{0}$, which was our claim.

We now invoke [RS19, Theorem 1.3], which is a quantitative version of the classical Runge approximation. In particular, fix $r_{0} > 0$ (smaller than $\rho_{0} > 0$ above) and let $\bar{h}_{i}$ be given by (5.4). By [RS19], there exists $C > 0$ and $\mu \geq 1$ (depending on $r_{0}$) such that for every $\varepsilon_{1} > 0$ and for every $\bar{h}_{i}$, we can find $h_{i}$ such that $Lh_{i} = 0$ in $B_{2}(0)$ and $h_{i}$ satisfies:

$$\|h_{i} - \bar{h}_{i}\|_{L^{2}(B_{3/2}r_{0}(p))} \leq \varepsilon_{1}\|\bar{h}_{i}\|_{W^{1,2}(B_{2r_{0}}(p))}$$

and

$$\|h_{i}\|_{H^{1/2}(\partial B_{2}(p))} \leq C\varepsilon_{1}^{-\mu}\|\bar{h}_{i}\|_{L^{2}(B_{3/2}r_{0}(p))}.$$

Because $h_{i}, \bar{h}_{i}$ satisfy the same equation in $B_{(3/2)r_{0}}(p)$, we can use [GT01, Theorems 8.17 and 8.32] to bound

$$\|\nabla(h_{i} - \bar{h}_{i})\|_{L^{\infty}(B_{r_{0}}(p))} \leq C\|h_{i} - \bar{h}_{i}\|_{L^{\infty}(B_{2r_{0}}(p))} \leq C\varepsilon_{1}^{2} \int_{B_{2r_{0}}(p)} |\nabla \bar{h}_{i}|^{2} + \bar{h}_{i}^{2} \leq C\varepsilon_{1}^{2},$$

where we have used the Lipschitz continuity of $A$ (5.5).
We can show that this last integral is independent of an equation in all of $B$ norm of its gradient. As the one above. This proves the upper bound on the supremum of $h$ to each $h_i$, which satisfies an equation and thus minimizes an energy, to $x_i$ which has the same boundary values and thus larger energy inside of $B_{2r_0}(p)$. The smallness condition on $\|\nabla h_i - e\|_2$ follows from this computation and the claim established in the previous paragraph. (First pick $\varepsilon_0$, which fixes $r_0$. Then pick $\varepsilon_1$ small enough in terms of $n, M, \lambda, \Lambda,$ and $r_0$.)

We have now fixed $\varepsilon_1, \varepsilon_0 > 0$ (depending on $n, M, \lambda, \Lambda,$ and 100^{-100}). Since $h_i$ satisfies an equation in all of $B_2(0)$ and $c \geq 0$, we invoke [GT01] Theorems 8.17 and 8.32] in order to estimate

$$\max \left\{ \int_{B_1(0)} |\nabla h_i|^2 \, dx, \sup_{\partial B_1(0)} |h_i|^2 \right\} \leq C \|h_i\|_{L^2(B_{4/3}(0))}^2 \leq C \varepsilon^{-2\mu} \int_{B_{3/2}(r_0(p)} \tilde{h}_i^2 \, dx.$$ 

We can show that this last integral is independent of $p, r_0$ by a similar comparison to $x_i$ as the one above. This proves the upper bound on the supremum of $h_i$ and on the $L^2$ norm of its gradient.

In the special case when $c(x) \equiv 0$, we see that $|h_i(0)| \leq C$ (because each $h_i$ satisfies the maximum principle in $B_1(0)$). Thus we may add a constant of absolute value at most $C$ to each $h_i$ to make sure that $h_i(0) = 0$ and not change any of the other arguments.

Finally, to get the lower bound on $\|h_i\|_{L^2(\partial B_1(0))}$, note that the gradient condition on $h_i$ implies $\text{osc}_{B_{3/4}(0)} h_i \geq C^{-1}r$ and invoke [GT01] Theorem 8.24] to get a lower bound on $\|h_i\|_{L^2(B_{7/8}(0))}$. Since $c \geq 0$, the quantitative solvability of the $L^2$-Dirichlet problem for these operators then gives the lower bound on $L^2(\partial B_1(0))$. Let us sketch the proof of this fact here: assume to the contrary and there exists $L_i \in \mathcal{L}(M, \lambda, \Lambda)$ with $A_i$ symmetric, $b_i \equiv 0$ and $c_i \geq 0$ and $\varphi_i \in L^2(\partial B_1(0))$ such that the solution $u_i$ to the boundary value problem $\nabla u_i = 0$ in $B_1(0)$ and $u_i = \varphi_i$ on $\partial B_1(0)$ satisfies

$$C \geq \|u_i\|_{W^{1,2}(B_1(0))} \geq \|u_i\|_{L^2(B_{7/8}(0))} \geq C^{-1},$$

but $\varphi_i \downarrow 0$ in $L^2(\partial B_1(0))$. Taking limits as in Corollary 2.3 we get that $u_i \to u_0$ in $W^{1,2}$, which solves $L_0 u_0 = 0$ in $B_1$, where $L_0 \in \mathcal{L}(M, \lambda, \Lambda)$. But $u_0 = 0$ on $\partial B_1(0)$, so invoking the maximum principle ($c_0 \geq 0$), we have $u_0 \equiv 0$. This contradicts the lower bound in the last displayed equation. $\square$

These conditions on the $h_i$ have the following geometric consequence, which we record here as a separate corollary:

**Corollary 5.2** (Existence of an “Origin”). Let $0 < r < 1$, let $p \in \mathbb{R}^n$, and for each $1 \leq i \leq n$, let $h_i \in C^{1,\alpha}(B_{2r}(p))$ satisfy $\|\nabla h_i - e\|_{L^\infty(B_r(p))} < 100^{-100}/n$. Then we can find a special point $q \in B_r(p)$ such that for all $\delta > 0$, there exists $\varepsilon_0 > 0$ such that

$$B(p, r) \cap \bigcap_{i=1}^n \{|h_i| < \varepsilon\} \subset B(q, \delta) \text{ for all } 0 < \varepsilon < \varepsilon_0.$$
Proof. By continuity of \( h_1, \ldots, h_n \), for any \( \delta > 0 \) there exists \( \varepsilon_0 > 0 \) such that

\[
B_r(p) \cap \bigcap_{i=1}^n \{ |h_i| < \varepsilon \} \subset N_\delta \left( \bigcap_{i=1}^n \{ |h_i| = 0 \} \right) \cap B_r(p),
\]

where \( N_\delta(E) = \{ x \in \mathbb{R}^n \mid \text{dist}(x, E) < \delta \} \) is the \( \delta \)-neighborhood of \( E \). Thus, it suffices to prove that

\[
B_r(p) \cap \left( \bigcap_{i=1}^n \{ |h_i| = 0 \} \right)
\]

is either empty or consists of a single point. Indeed, what we will show is that the map \( H : B_r(p) \rightarrow \mathbb{R}^n \) defined by \( H(x) = (h_1(x), h_2(x), \ldots, h_n(x)) \) is injective.

Let \( x, y \in B_r(p) \). We compute:

\[
|H(x) - H(y)|^2 = \left| \int_0^1 \frac{d}{dt} H(tx + (1-t)y) \, dt \right|^2 = \left| \int_0^1 DH(tx + (1-t)y) \cdot (x - y) \, dt \right|^2
\]

\[
= \sum_i \left| \int_0^1 \nabla h_i \cdot (x - y) \, dt \right|^2 = \sum_i \left| \int_0^1 e_i \cdot (x - y) + (\nabla h_i - e_i) \cdot (x - y) \, dt \right|^2
\]

\[
\geq \sum_i \left| (x_i - y_i) - (\sup_{B_r(p)} |\nabla h_i - e_i|)|x - y| \right|^2 \geq \sum_i \frac{1}{2} |x_i - y_i|^2 - \frac{3}{2} 100^{-200} |x - y|^2.
\]

Hence \( |H(x) - H(y)| \geq (1/2) |x - y| \). So the map \( H \) is injective inside of \( B_r(p) \)—actually, bi-Lipschitz—which means that all the \( h_i \) can vanish simultaneously at no more than a single point inside of \( B_r(p) \). The proof is complete. \( \square \)

Remark 5.3. Theorem \([5.1]\) relies only on interior estimates and \([RS19]\) Theorem 1.3]. The later quantitative Runge approximation theorem holds with appropriate constants as long as \( \Omega_1 \) and \( \Omega_2 \) are Lipschitz domains with \( \Omega_1 \subset \subset \Omega_2 \). Thus, Theorem \([5.1]\) holds if \( B_2(0) \) is replaced by a Lipschitz domain and \( B_{r_0}(p) \) is replaced by a smaller Lipschitz domain.

In the same vein, it is interesting to observe that Corollary \([5.2]\) holds when \( B_r(p) \) is replaced with any convex or star shaped domain, \( \Omega \). Indeed, as long as the domain is quasiconvex in the sense that there exists a \( 1 \leq K < \infty \) such that for every \( x, y \in \Omega \) there is a path \( \gamma \subset \Omega \) connecting \( x, y \) with length \( \ell(\gamma) \leq K|x - y| \), then Corollary \([5.2]\) holds with \( B_r(p) \) replaced by \( \Omega \) and replacing \( 100^{-100} \) by a smaller constant depending on \( K \).

Our main generic regularity result is a quantitative version of Theorem \([1.1]\) in the introduction. In view of Remark \([5.3]\) this result also holds if \( B_1 \) is replaced by a Lipschitz domain, \( \Omega \), and \( B_{1/2} \) is replaced by \( K \subset \subset \Omega \). Of course, the constants will then depend on the Lipschitz constants, \( K \) and \( \text{dist}(K, \partial \Omega) \). For the sake of simplicity, we prove the case where \( \Omega = B_1 \) and \( K = B_{1/2} \). The assumption that the operator \( L \) belongs to \( \mathcal{L}_{0,+}(M, \lambda, \Lambda) \) or \( \mathcal{L}_{0,\delta}(M, \lambda, \Lambda) \) with \( \delta \ll 1 \) will allow us to reduce to the case that \( L \) is purely second order by applying a ratio trick.\(^3\) See Lemma \([A.1]\).

\(^3\)The authors thank Alexander Logunov for suggesting this idea.
Theorem 5.4 (Main Theorem). Let \( L \) be an operator in \( \mathcal{L}_{0,+}(M,\lambda,\Lambda) \) or \( \mathcal{L}_{0,\delta}(M,\lambda,\Lambda) \) for some \( \delta \leq \delta_0(M,\lambda,\Lambda) \) sufficiently small. There exist constants \( \varepsilon_0 > 0 \) and \( C > 0 \), depending only on \( n, M, \lambda, \Lambda, \) and \( N_0 \), with the following property. If \( u \) is a solution of \( Lu = 0 \) in \( B_1(0) \) with \( N(u,0,1) \leq N_0 \) and \( \int_{\partial B_1} |u|^2 = 1 \), then for all \( 0 < \varepsilon < \varepsilon_0 \), there is \( v \in W^{1,2}(B_1(0)) \) such that \( C^{-1}\varepsilon < \|v - u\|_{L^2(\partial B_1)} < C\varepsilon \), \( Lv = 0 \) in \( B_1(0) \), \( v(0) = u(0) \), and \( \{v = 0\} \cap B_{1/2}(0) \) is a smooth submanifold.

Remark 5.5 (Perturbing Already Smooth Nodal Sets). One might ask what Theorem 5.4 can tell us when \( \{u = 0\} \) is already smooth. On one hand, Theorem 5.4 is trivially true when \( \{u = 0\} \) is smooth; merely set \( v = (1 \pm \varepsilon)u \) and the result follows. On the other hand, the perturbations constructed in proof of Theorem 5.4 are not of the form \((1 \pm \varepsilon)u\), in general. So Theorem 5.4 perhaps raises interesting questions even when \( S(u) = \emptyset \). However, these are outside the scope of this paper.

The idea of the proof, outlined in §1.1, is to cover \( B_{1/2}(0) \) with finitely many balls \( \{B(p_i,r_0/2)\}_{i=1}^J \), where \( p_i \in B_{1/2}(0) \) and \( r_0 \) is the radius guaranteed by Theorem 5.1. Because \( r_0 > 0 \) depends only \( n, M, \lambda, \) and \( \Lambda \), so does \( J \), the number of balls in the cover. Theorem 5.4 will be a consequence of the following lemma.

Lemma 5.6 (Local Smoothing). Let \( L \in \mathcal{L}_{0,0}(M,\lambda,\Lambda) \). There exists \( C > 0 \) (depending on \( n, M, \lambda, \Lambda, \) and \( N_0 \) appearing in the statement of Theorem 5.4) such that for any \( B = B(p_i,r_0/2) \) in a cover of \( B_{1/2}(0) \) and any \( 0 < \theta \leq \varepsilon_0 \), there exists \( v_{B,\theta} = v \in W^{1,2}(B_1(0)) \) with \( Lv = 0 \) in \( B_1(0) \) and \( v(0) = 0 \) such that

\[
(5.9) \quad 2C^{-1}\theta < \|v\|_{L^2(\partial B_1)} < C\theta/2, \quad \int_{B_1(0)} |\nabla v|^2 \leq C\theta^2, \quad \text{and} \quad \sup_{\partial B_1(0)} |v| < C\theta,
\]

and such that if \( Lw = 0 \) in \( B_1(0) \) with \( N(w,0,1) \leq 10N_0 \) and \( \int_{\partial B_1(0)} |w|^2 = 1 \), then \( \{w + v = 0\} \) is smooth in \( B = B(p_i,r_0/2) \).

Proof of Theorem 5.4 modulo Lemma 5.6. First we reduce to the case when \( L \in \mathcal{L}_{0,0} \). Let \( \delta_0 = \delta_0(M,\lambda,\Lambda) \) be the constant from Lemma A.1. Suppose \( Lu = -\text{div}(A(x)\nabla u) + cu \) belongs to \( \mathcal{L}_{0,+}(M,\lambda,\Lambda) \) or \( \mathcal{L}_{0,\delta}(M,\lambda,\Lambda) \) with \( \delta \leq \delta_0 \) and that \( c \neq 0 \). Lemma A.1 yields an operator \( \tilde{L} \in \mathcal{L}_{0,0}(M,\tilde{\lambda},\tilde{\Lambda}) \) and functions \( w \) and \( f \equiv u/w \) such that \( C^{-1} \leq w \leq C \), \( \tilde{L}f = 0 \) in \( B_1(0) \), \( N(f,0,1) \leq CN_0 \), and \( f = u \) on \( \partial B_1(0) \). If the theorem holds for purely second order operators, then we can perturb \( f \) to \( \tilde{f} \) which satisfies the relevant bounds, has smooth nodal set in \( B_{1/2}(0) \), and satisfies \( f(0) = \tilde{f}(0) \). Then a perturbation of \( u \) with the properties that we are looking for is given by \( v \equiv w\tilde{f} \). Since \( w \) is bounded above and away from zero and since \( w = 1 \) on \( \partial B_1 \), we see that \( v \) satisfies the desired conditions. Thus, without loss of generality, we may assume that \( L \in \mathcal{L}_{0,0}(M,\lambda,\Lambda) \).

Choose any minimal cover \( \bigcup_{i=1}^J B(p_i,r_0/2) \supset B_{1/2}(0) \), with \( r_0 \) given by Theorem 5.1. Fix \( \varepsilon < \varepsilon_0 \), with \( \varepsilon_0 \) given by Lemma 5.6. We consider the perturbation of \( u \) defined by

\[
(5.10) \quad v \equiv u + v_1 + v_2 + v_3 \ldots + v_J,
\]
where \( v_j = v_{B(p_j,r_0/2),\theta_j} \) are functions given by Lemma 5.6 applied with \( B = B(p_j,r_0/2) \) so that (5.9) holds with \( \theta_j > 0 \). The errors \( \theta_j \) will satisfy \( 1 > \theta_1 = \varepsilon \gg \theta_2 \gg \ldots \gg \theta_M > 0 \) and be chosen as follows: If we assume that \( \{u + v_1 + v_2 + \ldots + v_{n-1} = 0\} \) is smooth in \( \bigcup_{i=1}^{k-1} B(p_i,r_0/2) \), then

\[
(5.11) \quad \inf_{\bigcup_{i=1}^{k-1} B(p_i,r_0/2)} F(u + v_1 + v_2 + \ldots + v_{k-1}) \equiv \eta_k > 0, 
\]

where \( F(g) = g^2 + |\nabla g|^2 \) and \( \eta_k > 0 \) is some small number, which may not be uniform in any of the parameters. Since we are merely claiming that such an \( \eta_n \) exists, (5.11) follows simply from the continuity and non-vanishing of \( F(u + v_1 + v_2 + \ldots + v_{k-1}) \) on \( \bigcup_{i=1}^{k-1} B(p_i,r_0/2) \). Then we pick \( \theta_k \ll \eta_k \), so that \( \sup_{B(0)} F(v_k) \ll \eta_k \). In this way, we can guarantee that

\[
(5.12) \quad \inf_{\bigcup_{i=1}^{k-1} B(p_i,r_0/2)} F(u + v_1 + v_2 + \ldots + v_{k-1} + v_k) > \eta_k/2, 
\]

where we note that we now sum through \( v_k \) in (5.12). In the end, maybe shrinking the \( \theta_i \) (for \( i \geq 2 \)), we can guarantee that \( \|v_2 + \cdots + v_J\| < C^{-1} \varepsilon^2 \), which implies that

\[
C^{-1} \varepsilon < 2C^{-1} \varepsilon - C^{-1} \varepsilon^2 \leq \|v_1\|_{L^2(\partial B_1)} - \|v_2 + \cdots + v_J\|_{L^2(\partial B_1)} \\
\leq \|v_1 + v_2 + \cdots + v_J\|_{L^2(\partial B_1)} - \|v - u\|_{L^2(\partial B_1)} \\
\leq \|v_1\|_{L^2(\partial B_1)} + \|v_2 + \cdots + v_J\|_{L^2(\partial B_1)} - C^{-1} \varepsilon^2 + C \varepsilon/2 < C \varepsilon. 
\]

To finish the proof of the theorem, assuming Lemma 5.6 it suffices to show that for any \( k \leq J \), the function

\[
w_k \equiv u + v_1 + v_2 + \ldots + v_k 
\]

satisfies \( Lw_k = 0 \) in \( B_1(0) \), \( N(w_k,0,1) \leq (5 + \frac{5k}{2})N_0 \), \( 1 - C \varepsilon^2 \leq \int_{\partial B_1} |w_k|^2 \leq 1 + C \varepsilon^2 \), and \( \{w_k = 0\} \) is smooth in \( \bigcup_{i=1}^{k} B(p_i,r_0/2) \).

When \( k = 1 \), note that \( N(u,0,1) \leq N_0 \leq 10N_0 \) and \( \int_{\partial B_1} u^2 = 1 \) so Lemma 5.6 implies that \( \{u + v_1 = 0\} \) is smooth in \( B(p_1,r_0/2) \). Further, \( \int_{B_1(0)} |\nabla u_1|^2 \leq C \varepsilon^2 \), \( \sup_{\partial B_1(0)} |u_1| < C \varepsilon \). So as long as \( \varepsilon < \varepsilon_0 \) is small enough (depending only on \( C,J,N_0 \)), we can guarantee that \( N(u+u_1,0,1) \leq (1 + \frac{1}{J})N_0 \) and \( 1 - C \varepsilon^2 \leq \int_{\partial B_1} |u + v_1|^2 \leq 1 + C \varepsilon^2 \). Thus we have established the \( k = 1 \) case.

The arbitrary \( k \) case follows inductively: Lemma 5.6 tells us that \( \{u + v_1 + \ldots + v_k = 0\} \) is smooth in \( B(p_k,r_0/2) \). Here we have used the induction hypothesis which bounds the size of \( N(u + v_1 + \ldots + v_{k-1},0,1) \) and \( \int_{\partial B_1(0)} |u + v_1 + \ldots + v_{k-1}|^2 \). This integral is not equal to 1, but as long as it is close to 1, a straightforward rescaling argument allows us to apply Lemma 5.6. Furthermore, the arguments above (i.e. how we chose \( \theta_k \)) show

\[
\{u + v_1 + \ldots + v_k = 0\} \cap \left( \bigcup_{i=1}^{k-1} B(p_i,r_0/2) \right), \]
remains smooth. The bounds on the frequency $N(u + v_1 + \ldots + v_{k-1} + v_k, 0, 1)$ and also $\int_{\partial B_1} |u + v_1 + \ldots + v_{k-1} + v_k|^2$ follow as they did in the $k = 1$ case. Finally, $v(0) = u(0)$, since $v_i(0) = 0$ for each $i$. This completes our proof of Theorem 5.4 given Lemma 5.6. □

We turn to the proof of smoothing lemma, which requires both harmonic coordinates (Theorem 5.1, Corollary 5.2) and the gradient Lojasiewicz inequality (Theorem 1.3).

Proof of Lemma 5.6. Fix $\theta > 0$ and a ball $B = B(p, r_0/2)$. Instead of carrying around the index $i$, we will drop it, as all our estimates will be independent of the center of the ball. We will construct a function $v$, adapted to the larger ball $B(p, r_0)$ such that $L_v = 0$ in $B_1$, $v(0) = 0$, and $v$ satisfies (5.9), and such that if $Lw = 0$ in $B_1(0)$ with $N(w, 0, 1) \leq 10N_0$ and $\int_{\partial B_1(0)} |w|^2 = 1$, then \{w + v\} is smooth in $B(p, r_0/2)$.

Let $h_1, \ldots, h_n$ be the coordinate functions given by Theorem 5.1 adapted to the ball $B(p, r_0)$. That is to say, $Lh_i = 0$ in $B_1$ and

$$\sup_{x \in B_{r_0}(p)} \|\nabla h_i(x) - e_i\| < \frac{1}{n} 100^{-100}.$$ 

Furthermore, for all $i = 1, \ldots, n$,

$$\int_{B(0,1)} |\nabla h_i|^2 \leq C, \sup_{\partial B_1} |h_i| < C, \text{ and } C^{-1} r_0 \leq \|h_i\|_{L^2(\partial B_1(0))}.$$ 

Let $q$ be the “origin” given by Corollary 5.2, i.e., the unique point such that $h_i(q) = 0$ for all $i = 1, \ldots, n$. There may be no point at which all the $h_i$ vanish, in which case, let $q \in B(p, r_0/2)$ be arbitrary. Since $||\nabla h_i - 1|| < 100^{-100}$ in all of $B(p, r_0)$, there is a choice of sign $\pm$ such that

$$F(w \pm \theta h_1)(q) \equiv |\nabla(w \pm \theta h_1)|^2(q) + (w \pm \theta h_1)^2(q) > \theta^2/4.$$ 

Without loss of generality, assume that $F(w + \theta h_1)(q) > \theta^2/4$. Note that

$$N(w + \theta h_1, 0, 1) < \frac{5}{2}N_0 \text{ and } \int_{\partial B_1} |w + \theta h_1|^2 < 2N_0 + 10\theta C < \frac{5}{2}N_0,$$ 

as long as $\theta$ is small enough. By elliptic regularity, this implies that

$$\|w + \theta h_1\|_{C^{1,\alpha}(B(p, r_0))} \lesssim N_0 1,$$ 

which implies that there is some $\delta > 0$ (depending on $N_0$, $C$, $\theta$, $n$, $M$, $\lambda$, $\Lambda$, but not $w$) such that $F(w + \theta h_1)(y) > \theta^2/8, \forall y \in B(q, \delta)$.

Our perturbation, $v$, will be given by

$$v \equiv \theta h_1 + \varepsilon_1 h_1 + \varepsilon_2 h_2 + \ldots + \varepsilon_n h_n.$$ 

Since each $h_i(0) = 0$, it follows that $v(0) = 0$. Note we perturb by $h_1$ twice; our first perturbation was simply to create a neighborhood of $q$ in which there can be no singular points; we could have chosen to perturb by any solution of the PDE whose gradient did not vanish at $q$. Our subsequent perturbations will ensure that there are no singular points in the rest of $B(p, r_0/2)$, but must be chosen small enough so that we don’t reintroduce singular points into $B(q, \delta)$.
To that end, we will choose \( \theta \gg \varepsilon_1 \gg \varepsilon_2 \gg \ldots \gg \varepsilon_n \). Note, that as long as the \( \varepsilon_i \) are chosen small enough, the bounds on the norms of the \( h_i \) will imply that this perturbation satisfies the desired size constraints \( (5.9) \). Thus, it suffices to show that \( \{w + v = 0\} \) is a smooth manifold in \( B(p, r_0/2) \).

To see this (and explain how we choose the \( \varepsilon_i \)), we work inductively. Let \( \tilde{w} = w + \theta h_1 \).

Note, \( L(\tilde{w}) = 0 \) in \( B_1(0) \), \( N(\tilde{w}, 0, 1) < 10N_0 \), and \( \int_{\partial B_1} |\tilde{w}|^2 \approx 1+\theta \). Hence \( \tilde{w} \) satisfies the uniform gradient Lojasiewicz inequality (Theorem 1.3) in a \( \rho_1 \) neighborhood of its zero set. (Of course, \( \rho_1 \) will depend on \( \tilde{w} \), but that will not matter.) By continuity of \( F(\tilde{w}) \), we may pick \( \varepsilon_1 > 0 \) small so that \( F(\tilde{w})(x) < \varepsilon_1 \Rightarrow \text{dist}(x, \{F(\tilde{w}) = 0\}) < \rho_1 \).

(Again, the choice of \( \varepsilon_1 \) is not uniform, but it will not concern us.) By Theorem 1.3, if \( z \in S(\tilde{w} + \varepsilon_1 h_1) \), then

\[
(5.14) \quad c_2 \varepsilon_1 |h_1(z)| = c_2 |\tilde{w}(z)| \leq |\nabla \tilde{w}|^{c_1 N_0 / c_1 N_0 - 1} \lesssim N_0 |\nabla_{\tilde{w}}|^{c_1 N_0 / c_1 N_0 - 1} \lesssim N_0 \varepsilon_1^{c_1 N_0 / c_1 N_0 - 1},
\]

where \( c_1, c_2 \) are the constants from Theorem 1.3.

For the inductive step, let \( \tilde{w} = w + \theta h_1 + \varepsilon_1 h_1 + \ldots + \varepsilon_j h_j \), and we perturb by \( \varepsilon_{j+1} h_{j+1} \). By the inductive hypothesis

\[
S(\tilde{w}) \subset \bigcap_{i=1}^j \{|h_i| \leq c \varepsilon_i^{1/(c_1 N_0 - 1)}\},
\]

where \( c > 0 \) can depend on \( j, N_0, M, \lambda, \Lambda \). Again by the continuity of \( F(\tilde{w}) \) we can pick \( \varepsilon_{j+1} \) so small so that \( F(\tilde{w})(x) < \varepsilon_{j+1} \) implies that \( \text{dist}(x, \{F(\tilde{w}) = 0\}) < \rho_{j+1} \). Possibly shrinking \( \rho_{j+1} \) and recalling that the \( h_j \) are Lipschitz, we can also ensure that \( F(\tilde{w} + \varepsilon_{j+1} h_{j+1})(x_0) = 0 \) implies \( x_0 \in \bigcap_{i=1}^j \{|h_i| \leq 2 c \varepsilon_i^{1/(c_1 N_0 - 1)}\} \). Repeating the estimates in \( (5.14) \), we obtain

\[
S(\tilde{w} + \varepsilon_{j+1} h_{j+1}) \subset \bigcap_{i=1}^{j+1} \{|h_i| \leq 2 c \varepsilon_i^{1/(c_1 N_0 - 1)}\}.
\]

In the end, it follows that

\[
S(w + v) \subset \bigcap_{i=1}^n \{|h_i| \leq c \varepsilon_i^{1/(c_1 N_0 - 1)}\},
\]

where \( c > 1 \) is large but depends only on \( n, J, M, \lambda, \Lambda, \) and \( N_0 \). Letting \( \varepsilon_i \) for \( i = 1, \ldots, n \) be sufficiently small, we can guarantee by Corollary 5.2 that, with \( q \) and \( \delta \) as above,

\[
\bigcap_{i=1}^n \{|h_i| \leq c \varepsilon_i^{1/(c_1 N_0 - 1)}\} \subset B(q, \delta).
\]

However, we have already shown that \( F(w + \theta h_1)(y) > \eta^2/8 \) on \( B(q, \delta) \cap B(p, r_0/2) \). Shrinking the \( \varepsilon_i \) even more, as needed, we can guarantee that \( F(w + v)(y) > \theta^2/16 \) for all \( y \in B(q, \delta) \cap B(p, r_0/2) \). Thus, the singular set is empty: \( S(w + v) \cap B(p, r_0/2) = \emptyset \). \( \square \)
The following lemma and its application in the proof of Theorem 5.4 was suggested to us by Alexander Logunov.

**Lemma A.1 (The Ratio Trick).** There exists \( \delta_0 = \delta_0(M, \lambda, \Lambda) > 0 \) with the following property. Let \( L \in \mathcal{L}_{0,+}(M, \lambda, \Lambda) \) or \( L \in \mathcal{L}_{0}(M, \lambda, \Lambda) \) with \( \delta \leq \delta_0 \). If \( u \) solves \( Lu = 0 \) in \( B_1(0) \), then there is a unique solution, \( w \), to

\[
Lw = 0, \quad \text{in } B_1(0) \quad \text{and} \quad w = 1 \quad \text{on } \partial B_1(0),
\]

and a constant \( C = C(M, \Lambda, \lambda) > 0 \) such that \( C^{-1} \leq w \leq C \) in \( B_1(0) \). Furthermore,

\[
f \equiv \frac{u}{w}
\]

solves \( \tilde{L}f = 0 \) in \( B_1(0) \) for some operator \( \tilde{L} \in \mathcal{L}_{0,0}(\tilde{M}, \tilde{\lambda}, \tilde{\Lambda}) \) and constants \( \tilde{M}, \tilde{\lambda}, \text{ and } \tilde{\Lambda} \) depending on \( n, M, \lambda, \text{ and } \Lambda \). If, in addition, \( N(u, 0, 1) \leq N_0 \), then \( N(f, 0, 1) \leq C N_0 + 1 \) for some \( C = C(M, \Lambda, \lambda) > 0 \).

**Proof.** By assumption, we may write \( Lu \equiv -\text{div}(A(x)\nabla u) + cu \). If \( c \geq 0 \) or \( ||c||_{L^\infty} \) is small enough (depending on the ellipticity of \( A \)), then the Dirichlet problem is uniquely solvable for \( L \) in \( B_1(0) \). By boundary Schauder estimates (see e.g. [GT01, Theorem 8.33]), we have \( ||v||_{C^{1,\alpha}(B_0(0))} \leq K = K(M, \lambda, \Lambda) \). When \( c \geq 0 \) the lower bound on \( w \) comes from the strong maximum principle (see, [Eva10, Theorem 4, Section 6.4]). In the case that \( c \) has small \( L^\infty \) norm, the lower bound follows from a compactness argument (cf. Corollary 2.3) and the fact that if \( c = 0 \), then \( w = 1 \) solves (A.1).

It follows from some algebraic manipulation that \( f \) defined by (A.2) satisfies

\[-\text{div}(w^2 A \nabla f) = 0 \text{ in } B_1(0) \quad \text{and} \quad f = u \text{ on } \partial B_1(0).\]

From the upper and lower bound on \( w \) and the \( C^{1,\alpha} \) bound on \( w \), we see that \( \tilde{L}f = 0 \) with \( \tilde{L} \in \mathcal{L}_{0,0}(\tilde{M}, \tilde{\lambda}, \tilde{\Lambda}) \). Thus, it remains to establish the bound on the frequency of \( f \).

Suppose that \( N(u, 0, 1) \leq N_0 \), where the Almgren frequency function \( N \) is defined by (2.7). Since \( f = u \) on \( \partial B_1 \), the desired bound on \( N(f, 0, 1) \) amounts to bounding \( \int_{B_1(0)} |\nabla f|^2 \) by \( C N_0 \int_{\partial B_1(0)} u^2 \). Using the lower bounds on \( w \), we first estimate

\[
\int_{B_1(0)} |\nabla f|^2 \leq C \int_{B_1(0)} \frac{|\nabla u|^2}{w^2} + \frac{u^2 |\nabla w|^2}{w^4} \leq C N_0 \int_{\partial B_1(0)} u^2 + C ||w||^2_{C^{1,\alpha}} \int_{B_1(0)} u^2.
\]

To proceed, let \( \overline{u} \) denote the harmonic extension of \( u|_{\partial B_1(0)} \) in \( B_1(0) \). By the Poincaré inequality and harmonicity of \( \overline{u} \),

\[
\int_{B_1(0)} |u - \overline{u}|^2 \leq C \int_{B_1(0)} |\nabla (u - \overline{u})|^2 \implies \int_{B_1(0)} u^2 \leq 2 \int_{B_1} \overline{u}^2 + 8C \int_{B_1(0)} |\nabla u|^2 \leq 2 \int_{B_1} \overline{u}^2 + 8CN_0 \int_{\partial B_1(0)} u^2.
\]
Expanding \( u|_{\partial B_1(0)} = \sum_0^\infty c_i \phi_i \), where \( \{\phi_i\}_0^\infty \) are spherical harmonics of degree \( i \), we see that \( \overline{u}(r, \theta) = \sum c_i r^i \phi_i \). Thus,

\[
\int_{B_1} \overline{u}^2 = \sum_0^\infty \frac{1}{n+2i} c_i^2 \leq \sum_0^\infty (i+1)c_i^2 = \int_{B_1} |\nabla \overline{u}|^2 + \int_{\partial B_1} u^2 \\
\leq \int_{B_1} |\nabla u|^2 + \int_{\partial B_1} u^2 \leq (N_0 + 1) \int_{\partial B_1} u^2.
\]

□

REFERENCES


