HÖLDER CONNECTEDNESS AND PARAMETERIZATION OF ITERATED FUNCTION SYSTEMS

MATTHEW BADGER AND VYRON VELLIS

Abstract. We investigate the Hölder geometry of curves generated by iterated function systems (IFS) in a complete metric space. A theorem of Hata from 1985 asserts that every connected attractor of an IFS is locally connected and path-connected. In our primary result, we give a quantitative strengthening of Hata's theorem. We first prove that every connected attractor of an IFS is $(1/s)$-Hölder path-connected, where s is the similarity dimension of the IFS. We then show that every connected attractor of an IFS is parameterized by a $(1/\alpha)$-Hölder curve for all $\alpha > s$. At the endpoint, $\alpha = s$, a theorem of Remes from 1998 already established that connected self-similar sets in Euclidean space that satisfy the open set condition are parameterized by $(1/s)$-Hölder curves. In a secondary result, we show how to promote Remes' theorem to self-similar sets in complete metric spaces, but in this setting require the attractor to have positive s-dimensional Hausdorff measure in lieu of the open set condition. To close the paper, we determine sharp Hölder exponents of parameterizations in the class of connected self-affine Bedford-McMullen carpets and build parameterizations of self-affine sponges.

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1. Introduction

A special feature of one-dimensional metric geometry is the compatibility of intrinsic and extrinsic measurements of the length of a curve. Indeed, a theorem of Ważewski [Waż27] from the 1920s asserts that in a metric space a connected, compact set Γ admits

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a continuous parameterization of finite total variation (intrinsic length) if and only if the set has finite one-dimensional Hausdorff measure $\mathcal{H}^1$ (extrinsic length). In fact, any curve of finite length admits parameterizations $f : [0,1] \to \Gamma$, which are closed, Lipschitz, surjective, degree zero, constant speed, essentially two-to-one, and have total variation equal to $2\mathcal{H}^1(\Gamma)$; see Alberti and Ottolini [AO17, Theorem 4.4]. Unfortunately, this phenomenon—compatibility of intrinsic and extrinsic measurements of size—breaks down for higher-dimensional curves. While every curve parameterized by a continuous map of finite s-variation has finite s-dimensional Hausdorff measure $\mathcal{H}^s$, for each real-valued dimension $s > 1$ there exist curves with $0 < \mathcal{H}^s(\Gamma) < \infty$ that cannot be parameterized by a continuous map of finite s-variation; e.g. see the "Cantor ladders" in [BNV19, §9.2]. Beyond a small zoo of examples, there does not yet exist a comprehensive theory of curves of dimension greater than one. Partial investigations on Hölder geometry of curves from a geometric measure theory perspective include [MM93], [MM00], [RZ16], [BV19], [BNV19], and [BZ19] (also see [Bad19]). For example, in [BNV19] with Naples, we established a Ważewski-type theorem for higher-dimensional curves under an additional geometric assumption (flatness), which is satisfied e.g. by von Koch snowflakes with small angles. The fundamental challenge is to develop robust methods to build good parameterizations.

Two well-known examples of higher-dimensional curves with Hölder parameterizations are the von Koch snowflake and the square (a space-filling curve). A common feature is that both examples can be viewed as the attractors of iterated function systems (IFS) in Euclidean space that satisfy the open set condition (OSC); for a quick review of the theory of IFS, see §2. Remes [Rem98] proved that this observation is generic in so far as every connected self-similar set in Euclidean space of Hausdorff dimension $s \geq 1$ satisfying the OSC is a $(1/s)$-*Hölder curve*, i.e. the image of a continuous map $f : [0,1] \to \mathbb{R}^n$ satisfying

$$|f(x) - f(y)| \leq H|x-y|^{1/s} \quad \text{for all } x, y \in [0,1]$$

for some constant $H < \infty$. As an immediate consequence, for every integer $n \geq 2$ and real number $s \in (1, n]$, we can easily generate a plethora of examples of $(1/s)$-Hölder curves in $\mathbb{R}^n$ with $0 < \mathcal{H}^s(\Gamma) < \infty$. See Figure 1. However, with the view of needing

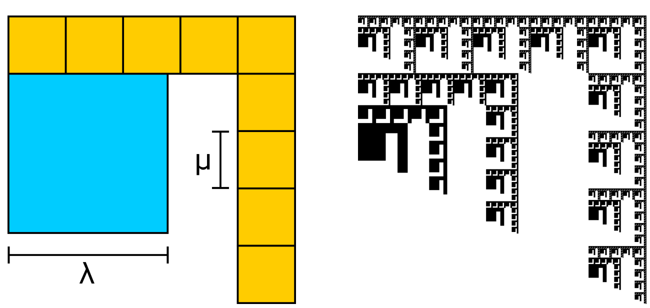

FIGURE 1. First and fourth iterations generating a self-similar $(1/s)$-Hölder curve Γ in $\mathbb{R}^2$ with $0 < \mathcal{H}^s(\Gamma) < \infty$; adjusting $\lambda \in [0, 1-\mu]$ and $\mu = 1/k$ (where $k \geq 2$ is an integer) yields examples of every dimension $s \in (1, 2]$.

a better theory of curves of dimension greater than one, we may ask whether Remes' method is flexible enough to generate Hölder curves under less stringent requirements, e.g. can we parameterize self-similar sets in metric spaces or arbitrary connected IFS? The naive answer to this question is no, in part because measure-theoretic properties of IFS attractors in general metric or Banach spaces are less regular than in Euclidean space (see Schief [Sch96]). Nevertheless, combining ideas from Remes [Rem98] and Badger-Vellis [BV19] (or Badger-Schul [BS16]), we establish the following pair of results in the general metric setting. We emphasize that Theorems 1.1 and 1.2 do not require the IFS to be generated by similarities nor do they require the OSC. In the statement of the theorems, extending usual terminology for self-similar sets, we say that the *similarity dimension* of an IFS generated by contractions $\mathcal{F}$ is the unique number s such that

$$\sum_{\phi\in\mathcal{F}}(\operatorname{Lip}\phi)^s = 1, \tag{1.1}$$

where $\operatorname{Lip}\phi = \sup_{x\neq y}\operatorname{dist}(\phi(x),\phi(y))/\operatorname{dist}(x,y)$ is the Lipschitz constant of ϕ.

Theorem 1.1 (Hölder connectedness)**.** *Let $\mathcal{F}$ be an IFS over a complete metric space; let s be the similarity dimension of $\mathcal{F}$. If the attractor $K_{\mathcal{F}}$ is connected, then every pair of points is connected in $K_{\mathcal{F}}$ by a $(1/s)$-Hölder curve.*

Theorem 1.2 (Hölder parameterization)**.** *Let $\mathcal{F}$ be an IFS over a complete metric space; let s be the similarity dimension of $\mathcal{F}$. If the attractor $K_{\mathcal{F}}$ is connected, then $K_{\mathcal{F}}$ is a $(1/\alpha)$-Hölder curve for every $\alpha > s$.*

Early in the development of fractals, Hata [Hat85] proved that if the attractor $K_{\mathcal{F}}$ of an IFS over a complete metric space X is connected, then $K_{\mathcal{F}}$ is locally connected and path-connected. By the Hahn-Mazurkiewicz theorem, it follows that if $K_{\mathcal{F}}$ is connected, then $K_{\mathcal{F}}$ is a *curve*, i.e. $K_{\mathcal{F}}$ the image of a continuous map from $[0,1]$ into X. Theorems 1.1 and 1.2, which are our main results, can be viewed as a quantitative strengthening of Hata's theorem. We prove the two theorems directly, in §3, without passing through Hata's theorem.

Roughly speaking, to prove Theorem 1.1, we embed the attractor $K_{\mathcal{F}}$ into ℓ_∞ and then construct a $(1/s)$-Hölder path between a given pair of points as the limit of a sequence of piecewise linear paths, mimicking the usual parameterization of the von Koch snowflake. Although the intermediate curves live in ℓ_∞ and not necessarily in $K_{\mathcal{F}}$, each successive approximation becomes closer to $K_{\mathcal{F}}$ in the Hausdorff metric so that the final curve is entirely contained in the attractor. Building the sequence of intermediate piecewise linear paths is a straightforward application of connectedness of an abstract word space associated to the IFS. The essential point to ensure the limit map is Hölder is to estimate the growth of the Lipschitz constants of the intermediate maps (see §2.2 for an overview). Condition (1.1) gives us a natural way to control the growth of the Lipschitz constants, and thus, the similarity dimension determines the Hölder exponent of the limiting map (see §3). A similar technique allows us to parameterize the whole attractor of an IFS *without branching* by a $(1/s)$-Hölder arc (see §4).

To prove Theorem 1.2, we view the attractor $K_{\mathcal{F}}$ as the limit of a sequence of metric trees $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \cdots$ whose edges are $(1/s)$-Hölder curves. Using condition (1.1), one can easily show that

$$S_\alpha := \sup_n \sum_{E \in \mathcal{T}_n} (\operatorname{diam} E)^\alpha < \infty \quad \text{for all } \alpha > s. \tag{1.2}$$

We then prove (generalizing a construction from [BV19, §2]) that (1.2) ensures $K_{\mathcal{F}}$ is a $(1/\alpha)$-Hölder curve for all $\alpha > s$. Unfortunately, because the constants S_α in (1.2) diverge as $\alpha \downarrow s$, we cannot use this method to obtain a Hölder parameterization at the endpoint. We leave the question of whether or not one can always take $\alpha = s$ in Theorem 1.2 as an open problem. The central issue is find a good way to control the growth of Lipschitz or Hölder constants of intermediate approximations for connected IFS *with branching*.

For self-similar sets with positive $\mathcal{H}^s$ measure, we can build Hölder parameterizations at the endpoint in Theorem 1.2. The following theorem should be attributed to Remes [Rem98], who established the result for self-similar sets in Euclidean space, where the condition $\mathcal{H}^s(K_{\mathcal{F}}) > 0$ is equivalent to the OSC (see Schief [Sch94]). In metric spaces, it is known that $\mathcal{H}^s(K_{\mathcal{F}}) > 0$ implies the (strong) open set condition, but not conversely (see Schief [Sch96]). A key point is that self-similar sets $K_{\mathcal{F}}$ with positive $\mathcal{H}^s$ measure are necessarily *Ahlfors s-regular*, i.e. $r^s \lesssim \mathcal{H}^s(K_{\mathcal{F}} \cap B(x,r)) \lesssim r^s$ for all balls $B(x,r)$ centered on $K_{\mathcal{F}}$ with radius $0 < r \lesssim \operatorname{diam} K_{\mathcal{F}}$. This fact is central to Remes' method for parameterizing self-similar sets with branching. See §5 for details.

Theorem 1.3 (Hölder parameterization for self-similar sets)**.** *Let $\mathcal{F}$ be an IFS over a complete metric space that is generated by similarities; let s be the similarity dimension of $\mathcal{F}$. If the attractor $K_{\mathcal{F}}$ is connected and $\mathcal{H}^s(K_{\mathcal{F}}) > 0$, then $K_{\mathcal{F}}$ is a $(1/s)$-Hölder curve.*

As a case study, in §6, to further illustrate the results above, we determine the sharp Hölder exponents in parameterizations of connected self-affine Bedford-McMullen carpets. We also build parameterizations of connected self-affine sponges in $\mathbb{R}^n$ (see Corollary 6.7). Of some note, the best Hölder exponent in parameterizations of a self-affine carpet can exceed 2. See Figure 2.

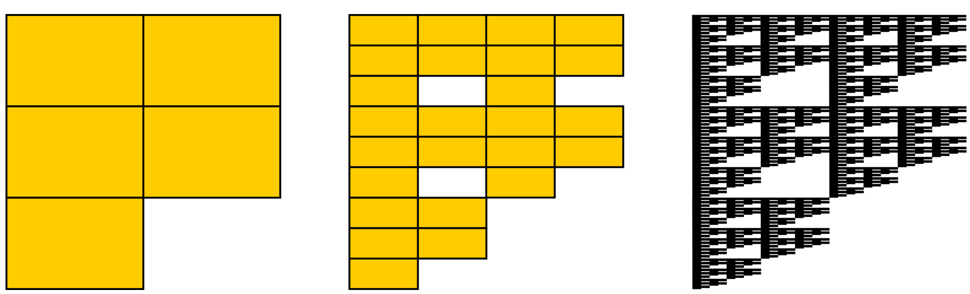

FIGURE 2. First, second, and fifth iterations of a Bedford-McMullen carpet Σ that is a self-affine $(1/s)$-Hölder curve (with $\mathcal{H}^s(\Sigma) = 0$) precisely when $s \geq \log_2(5) > 2$.

Theorem 1.4. *Let $\Sigma \subset [0,1]^2$ be a connected Bedford-McMullen carpet (see §6).*

- *If Σ is a line, then Σ is (trivially) a 1-Hölder curve.*
- *If Σ is the square, then Σ is (well-known to be) a $(1/2)$-Hölder curve.*
- *Otherwise, Σ is a $(1/s)$-Hölder curve, where s is the similarity dimension of Σ.*

The Hölder exponents above are sharp, i.e. they cannot be increased.

2. Preliminaries

2.1. **Iterated function systems.** Let X be a complete metric space. A *contraction* in X is a Lipschitz map $\phi : X \to X$ with Lipschitz constant $\operatorname{Lip}\phi < 1$, where

$$\operatorname{Lip}\phi := \sup_{x\neq y} \frac{\operatorname{dist}(\phi(x),\phi(y))}{\operatorname{dist}(x,y)} \in [0,\infty]. \tag{2.1}$$

An *iterated function system (IFS)* $\mathcal{F}$ is a finite collection of contractions in X. We say that $\mathcal{F}$ is *trivial* if $\operatorname{Lip}\phi = 0$ for every $\phi \in \mathcal{F}$; otherwise, we say that $\mathcal{F}$ is *non-trivial*. The *similarity dimension* $\text{s-dim}(\mathcal{F})$ of $\mathcal{F}$ is the unique number s such that

$$\sum_{\phi\in\mathcal{F}} (\operatorname{Lip}\phi)^s = 1, \tag{2.2}$$

with the convention $\text{s-dim}(\mathcal{F}) = 0$ whenever $\mathcal{F}$ is trivial. Iterated function systems were introduced by Hutchinson [Hut81] and encode familiar examples of fractal sets such as the Cantor ternary set, Sierpiński carpet, and Sierpiński gasket. For an extended introduction to IFS, see Kigami's *Analysis on Fractals* [Kig01]. Hutchinson's original paper as well as Hata's paper [Hat85] are gems in geometric analysis and excellent introductions to the subject in their own right.

Theorem 2.1 (Hutchinson [Hut81])**.** *If $\mathcal{F}$ is an IFS over a complete metric space, then there exists a unique compact set $K_{\mathcal{F}}$ in X (the* attractor *of $\mathcal{F}$) such that*

$$K_{\mathcal{F}} = \bigcup_{\phi\in\mathcal{F}} \phi(K_{\mathcal{F}}). \tag{2.3}$$

Furthermore, if $s = \text{s-dim}(\mathcal{F})$, then $\mathcal{H}^s(K_{\mathcal{F}}) \leq (\operatorname{diam} K_{\mathcal{F}})^s < \infty$ and $\dim_H(K_{\mathcal{F}}) \leq s$.

Above and below, the *s-dimensional Hausdorff measure* $\mathcal{H}^s$ on a metric space is the Borel regular outer measure defined by

$$\mathcal{H}^s(E) = \lim_{\delta\downarrow 0} \inf\left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s : E \subset \bigcup_{i=1}^{\infty} E_i,\ \sup_i \operatorname{diam} E_i \leq \delta \right\} \quad \text{for all } E \subset X. \tag{2.4}$$

The *Hausdorff dimension* $\dim_H(E)$ of a set E in X is the unique number given by

$$\dim_H(E) := \inf\{\alpha \in [0,\infty) : \mathcal{H}^\alpha(E) < \infty\} = \sup\{\beta \in [0,\infty) : \mathcal{H}^\beta(E) > 0\}. \tag{2.5}$$

For background on the fine properties of Hausdorff measures, Hausdorff dimension, and related elements of geometric measure theory, see Mattila's *Geometry of Sets and Measures in Euclidean Spaces* [Mat95].

We say that an IFS $\mathcal{F}$ over a metric space X satisfies the *open set condition (OSC)* if there exists an open set $U \subset X$ such that

$$\phi(U) \subset U \quad \text{and} \quad \phi(U) \cap \psi(U) = \emptyset \quad \text{for every } \phi, \psi \in \mathcal{F} \text{ with } \phi \neq \psi. \tag{2.6}$$

If there exists an open set $U \subset X$ satisfying (2.6), and in addition, $K_{\mathcal{F}} \cap U \neq \emptyset$, then we say that $\mathcal{F}$ satisfies the *strong open set condition (SOSC)*. We say that the attractor $K_{\mathcal{F}}$ of an IFS $\mathcal{F}$ over X is *self-similar* if each $\phi \in \mathcal{F}$ is a *similarity*, i.e. there exists a constant $0 \leq L_\phi < 1$ such that

$$\operatorname{dist}(\phi(x), \phi(y)) = L_\phi \operatorname{dist}(x, y) \quad \text{for all } x, y \in X. \tag{2.7}$$

Theorem 2.2 (Schief [Sch94], [Sch96]). *Let $K_{\mathcal{F}}$ be a self-similar set in X; let $s =$ s-dim$(\mathcal{F})$. If X is a complete metric space, then*

$$\mathcal{H}^s(K_{\mathcal{F}}) > 0 \Rightarrow \text{SOSC} \Rightarrow \dim_H(K_{\mathcal{F}}) = s. \tag{2.8}$$

If $X = \mathbb{R}^n$, then

$$\mathcal{H}^s(K_{\mathcal{F}}) > 0 \Leftrightarrow \text{SOSC} \Leftrightarrow \text{OSC} \Rightarrow \dim_H(K_{\mathcal{F}}) = s \tag{2.9}$$

Moreover, the implications above are the best possible (unlisted arrows are false).

Given a metric space X, a set $E \subset X$, and radius $\rho > 0$, let $N(E, \rho)$ denote the maximal number of disjoint closed balls with center in E and radius ρ. Following Larman [Lar67], X is called a *β-space* if for all $0 < \beta < 1$ there exist constants $1 \leq N_\beta < \infty$ and $r_\beta > 0$ such that $N(B, \beta r) \leq N_\beta$ for every open ball B of radius $0 < r \leq r_\beta$.

Theorem 2.3 (Stella [Ste92]). *Let $K_{\mathcal{F}}$ be a self-similar set in X; let $s =$ s-dim$(\mathcal{F})$. If X is a complete β-space, then*

$$\text{SOSC} \Rightarrow \mathcal{H}^s(K_{\mathcal{F}}) > 0. \tag{2.10}$$

The following pair of lemmas are easy exercises, whose proofs we leave for the reader.

Lemma 2.4. *Let $K_{\mathcal{F}}$ be a self-similar set in X; let $s =$ s-dim$(\mathcal{F})$. If $\mathcal{H}^s(K_{\mathcal{F}}) > 0$, then $K_{\mathcal{F}}$ is Ahlfors s-regular, i.e. there exists a constant $1 \leq C < \infty$ such that*

$$C^{-1} r^s \leq \mathcal{H}^s(K_{\mathcal{F}} \cap B(x, r)) \leq C r^s \quad \text{for all } x \in K_{\mathcal{F}} \text{ and } 0 < r \leq \operatorname{diam} K_{\mathcal{F}}. \tag{2.11}$$

Lemma 2.5. *Let $\mathcal{F}$ be an IFS over a complete metric space. If $K_{\mathcal{F}}$ is connected, $\operatorname{diam} K_{\mathcal{F}} > 0$, and $\phi \in \mathcal{F}$ has $\operatorname{Lip}(\phi) = 0$, then $K_{\mathcal{F}}$ agrees with the attractor of $\mathcal{F} \setminus \{\phi\}$.*

2.2. **Hölder parameterizations.** Let $s \geq 1$, let X be a metric space, and let $f : [0, 1] \to X$. We define the *s-variation* of f (*over* $[0, 1]$) by

$$\|f\|_{s\text{-var}} := \left(\sup_{\mathcal{P}} \sum_{I \in \mathcal{P}} (\operatorname{diam} f(I))^s \right)^{1/s} \in [0, +\infty], \tag{2.12}$$

where the supremum ranges over all finite interval partitions $\mathcal{P}$ of $[0, 1]$. Here and below a *finite interval partition* of an interval I is a collection of (possibly degenerate) intervals

$\{J_1,\dots,J_k\}$ that are mutually disjoint with $I=\bigcup_{i=1}^k J_i$. We say that the map f is $(1/s)$-*Hölder continuous* provided that the associated $(1/s)$-*Hölder constant*

$$\mathrm{Höld}_{1/s}(f) := \sup_{x\neq y} \frac{\mathrm{dist}(f(x),f(y))}{|x-y|^{1/s}} < \infty. \tag{2.13}$$

By now, the following connection between continuous maps of finite s-variation and $(1/s)$-Hölder continuous maps is a classic exercise; for a proof and some historical remarks, see Friz and Victoir's *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications* [FV10, Chapter 5]. Although, we do not invoke Lemma 2.6 directly below, behind the scenes many estimates that we carry out are motivated by trying to bound a discrete s-variation adapted to finite trees that we used in [BNV19, §4].

Lemma 2.6 ([FV10, Proposition 5.15])**.** *Let $s\geq 1$ and let $f:[0,1]\to X$ be continuous.*

1. *If f is $(1/s)$-Hölder, then $\|f\|_{s\text{-}var}\leq \mathrm{Höld}_{1/s} f$.*
2. *If $\|f\|_{s\text{-}var}<\infty$, then there exists a continuous surjection $\psi:[0,1]\to[0,1]$ and a $(1/s)$-Hölder map $F:[0,1]\to X$ such that $f=F\circ\psi$ and $\mathrm{Höld}_{1/s} F\leq \|f\|_{s\text{-}var}$.*

The standard method to build a Hölder parameterization of a curve in a Banach space that we employ below is to exhibit the curve as the pointwise limit of a sequence of Lipschitz curves with controlled growth of Lipschitz constants. We will use this principle frequently, and also on one occasion in §3, the following extension where the intermediate maps are Hölder continuous.

Lemma 2.7. *Let $1\leq t<s$, $M>0$, $0<\xi_1\leq\xi_2<1$, $\alpha>0$, $\beta>0$, and $j_0\in\mathbb{Z}$. Let $(X,|\cdot|)$ be a Banach space. Suppose that ρ_j $(j\geq j_0)$ is a sequence of scales and $f_j:[0,M]\to X$ $(j\geq j_0)$ is a sequence of $(1/t)$-Hölder maps satisfying*

1. *$\rho_{j_0}=1$ and $\xi_1\rho_j\leq\rho_{j+1}\leq\xi_2\rho_j$ for all $j\geq j_0$,*
2. *$|f_j(x)-f_j(y)|\leq A_j|x-y|^{1/t}$ for all $j\geq j_0$, where $A_j\leq\alpha\rho_j^{1-s/t}$, and*
3. *$|f_j(x)-f_{j+1}(x)|\leq B_j$ for all $j\geq j_0$, where $B_j\leq\beta\rho_j$.*

Then f_j converges uniformly to a map $f:[0,M]\to X$ such that

$$|f(x)-f(y)|\leq H|x-y|^{1/s}\quad \textit{for all } x,y\in[0,M],$$

where H is a finite constant depending on at most $\max(M,M^{-1})$, ξ_1, ξ_2, α, and β. In particular, we may take

$$H=\frac{\alpha}{\xi_1}\max(1,M)+\frac{2\beta}{\xi_1(1-\xi_2)}\max(1,M^{-1}). \tag{2.14}$$

Proof. The statement and proof in the case $t=1$ is written in full detail in [BNV19, Lemma B.1]. The proof of the general case follows *mutatis mutandis.* □

3. Hölder connectedness of IFS attractors

In this section, we first prove Theorem 1.1, and afterwards, we derive Theorem 1.2 as a corollary. To that end, for the rest of this section, fix an IFS $\mathcal{F}=\{\phi_1,\dots,\phi_k\}$ over a complete metric space (X, d) whose attractor $K := K_{\mathcal{F}}$ is connected and has positive diameter. Set $s := \text{s-dim}(\mathcal{F})$, and for each $i \in \{1, \dots, k\}$, set $L_i := \mathrm{Lip}(\phi_i)$. By Lemma 2.5, we may assume without loss of generality that

$$0 < L_1 \leq \cdots \leq L_k < 1. \tag{3.1}$$

By definition of the similarity dimension, we have $L_1^s + \cdots + L_k^s = 1$.

3.1. **Words.** Define the *alphabet* $A = \{1, \dots, k\}$. Let $A^n = \{i_1 \cdots i_n : i_1, \dots, i_n \in A\}$ denote the set of *words* in A and of *length* n. Also let $A^0 = \{\epsilon\}$ denote the set containing the *empty word* ϵ of length 0. Let $A^* = \bigcup_{n \geq 0} A^n$ denote the set of *finite words* in A. Given any finite word $w \in A^*$ and length $n \in \mathbb{N}$, we assign

$$A_w^* := \{u \in A^* : u = wv\} \quad \text{and} \quad A_w^n = \{wv \in A_w^* : |wv| = n\}. \tag{3.2}$$

The set A_w^* can be viewed in a natural way as a tree with root at w. We also let $A^{\mathbb{N}}$ denote the set of *infinite words* in A. Given an infinite word $w = i_1 i_2 \cdots \in A^{\mathbb{N}}$ and integer $n \geq 0$, we define the *truncated word* $w(n) = i_1 \cdots i_n$ with the convention that $w(0) = \epsilon$.

We now organize the set of finite words in A, according to the Lipschitz norms of the associated contractions. For each word $w = i_1 \cdots i_n \in A^*$, define the map

$$\phi_w := \phi_{i_1} \circ \cdots \circ \phi_{i_n} \tag{3.3}$$

and the weight

$$L_w := L_{i_1} \cdots L_{i_n}. \tag{3.4}$$

By convention, for the empty word, we assign $\phi_\epsilon := \mathrm{Id}_X$ and $L_\epsilon := 1$. For all $w \in A^*$, define K_w to be the image of the attractor under ϕ_w,

$$K_w := \phi_w(K). \tag{3.5}$$

Note that $L_{uv} = L_u L_v$ for every pair of words u and v, where uv denotes the concatenation of u followed by v. For each $\delta \in (0, 1)$, define

$$A^*(\delta) := \{i_1 \cdots i_n \in A^* : n \geq 1 \text{ and } L_{i_1} \cdots L_{i_n} < \delta \leq L_{i_1} \cdots L_{i_{n-1}}\} \tag{3.6}$$

with the convention $L_1 \cdots L_{i_{n-1}} = 1$ if $n = 1$. Also define $A^*(1) := \{\epsilon\}$. Finally, given any finite word $w \in A^*$ and length $n \in \mathbb{N}$, set $A_w^*(\delta) := A_w^* \cap A^*(\delta)$ and $A_w^n(\delta) := A_w^n \cap A^*(\delta)$.

Lemma 3.1. *Given finite words $w \in A^*$ and $w' = wi_1 \cdots i_n$ and a number $L_{w'} < \delta \leq L_w$, there exists a unique finite word $u = wi_1 \cdots i_m$ ($m \leq n$) such that $u \in A_w^*(\delta)$.*

Proof. Existence of u follows from the fact that the sequence $a_n = L_{wi_1 \cdots i_n}$ is decreasing. Uniqueness of u follows from the fact that if $wi_1 \cdots i_m \in A_w^*(\delta)$, then for every $l < m$, $L_{wi_1 \cdots i_l} \geq \delta$, whence $wi_1 \cdots i_l \notin A_w^*(\delta)$. □

Lemma 3.2. *For all $0 < r < R < 1$ and all $w \in A^*(R)$,*

$$L_1^s (R/r)^s < \operatorname{card} A_w^*(r) < L_1^{-s} (R/r)^s. \tag{3.7}$$

Proof. Fix $0 < r < R < 1$ and $w \in A^*(R)$. Then $L_w < R \le L_w/L_1$, and similarly, for all $wu \in A^*_w(r)$, we have $L_{wu} < r \le L_{wu}/L_1$. Choose any $N \in \mathbb{N}$ sufficiently large so that $L_{wv} < r$ for all $wv \in A^N_w$. By Lemma 3.1, for each $wv \in A^N_w$, there exists a unique $wu \in A^*_w(r)$ such that $wv = wuv'$. Because $L_{wu} = L_w L_u$ and $L_1^s + \cdots + L_k^s = 1$,

$$L_1^s r^s(\operatorname{card} A^*_w(r)) \le \sum_{wu \in A^*_w(r)} L_w^s L_u^s = \sum_{wu \in A^*_w(r)} \sum_{wuv \in A^N_{wu}} L_w^s L_{uv}^s = \sum_{wuv \in A^N_w} L^s_{wuv} = L^s_w < R^s.$$

Similarly,

$$r^s(\operatorname{card} A^*_w(r)) > \sum_{wu \in A^*_w(r)} L_w^s L_u^s = \sum_{wu \in A^*_w(r)} \sum_{wuv \in A^N_{wu}} L_w^s L_{uv}^s = \sum_{wuv \in A^N_w} L^s_{wuv} = L^s_w \ge L_1^s R^s$$

and the claim follows. □

3.2. **Hölder connectedness (Proof of Theorem 1.1).**

Lemma 3.3 (chain lemma)**.** *Assume that $K_{\mathcal{F}}$ is connected. Let $w \in A^*$ and $0 < \delta < L_w$. If $x, y \in K_w$, then there exist distinct words $w_1, \dots, w_n \in A^*_w(\delta)$ such that $x \in K_{w_1}$, $y \in K_{w_n}$, and $K_{w_i} \cap K_{w_{i+1}} \neq \emptyset$ for all $i \in \{1, \dots, n-1\}$.*

Proof. We first remark that $K_w = \bigcup_{u \in A^*_w(\delta)} K_u$ by Lemma 3.1. Define

$$E_1 := \{u \in A^*_w(\delta) : x \in K_u\}.$$

Assuming we have defined $E_1, \dots, E_i \subset A^*_w(\delta)$ for some $i \in \mathbb{N}$, define

$$E_{i+1} := \{u \in A^*_w(\delta) \setminus E_i : K_u \cap K_v \neq \emptyset \text{ for some } v \in E_i\}.$$

Because K_w is connected (since $K_{\mathcal{F}}$ is connected), if $\bigcup_{i=1}^j E_i \neq A^*_w(\delta)$, then $E_{j+1} \neq \emptyset$. Since $A^*_w(\delta)$ is finite, it follows that $\bigcup_{i=1}^N E_i = A^*_w(\delta)$ for some $N \in \mathbb{N}$.

Choose a word $v \in A^*_w(\delta)$ such that $y \in K_v$. Then $v \in E_n$ for some $1 \le n \le N$. Label $v =: w_n$. By design of the sets E_i, we can find a chain of distinct words $w_1, \dots, w_n$ with $K_{w_i} \cap K_{w_{i+1}}$ for all $1 \le i \le n-1$. Finally, $x \in K_{w_1}$, because $w_1 \in E_1$. □

Theorem 1.1 is a special case of the following sharper result (take w to be the empty word). Recall that a metric space (X, d) is *quasiconvex* if any pair of points x and y can be joined by a Lipschitz curve $f : [0,1] \to X$ with $\operatorname{Lip}(f) \lesssim_X d(x,y)$. By analogy, the following proposition may be interpreted as saying that connected attractors of IFS are "$(1/s)$-Hölder quasiconvex".

Proposition 3.4. *For any $w \in A^*$ and $x, y \in K_w$, there exists a $(1/s)$-Hölder continuous map $f : [0, L^s_w] \to K_w$ with $f(0) = x$, $f(L^s_w) = y$, and $\operatorname{Höld}_{1/s} f \lesssim_{s,L_1} \operatorname{diam} K$.*

Proof. By rescaling the metric on X, we may assume without loss of generality that $\operatorname{diam} K = 1$. Furthermore, it suffices to prove the proposition for $w = \epsilon$ and $K_w = K$. For the general case, fix $w \in A^*$ and $x, y \in K_w$. Choose $x', y' \in K$ such that $\phi_w(x') = x$ and $\phi_w(y') = y$. Define

$$\zeta_w : [0, L^s_w] \to [0,1], \qquad \zeta_w(t) = (L_w)^{-s} t \quad \text{for all } t \in [0, L^s_w].$$

If the proposition holds for $w = \epsilon$, then there exists a $(1/s)$-Hölder map $g : [0,1] \to K$ with $g(0) = x'$, $g(1) = y'$, and $\mathrm{Höld}_{1/s}\, g \lesssim_{s,L_1} 1$. Then the map $f \equiv \phi_w \circ g \circ \zeta_w : [0, L_w^s] \to K_w$ plainly satisfies $f(0) = x$ and $f(L_w^s) = y$. Moreover, for any $p, q \in [0, L_w^s]$,

$$d(f(p), f(q)) \leq L_w\, d(g(\zeta_w(p)), g(\zeta_w(q))) \lesssim_{s,L_1} L_w|\zeta_w(p) - \zeta_w(q)|^{1/s} = |p - q|^{1/s}.$$

Thus, $\mathrm{Höld}_{1/s}(f) \lesssim_{s,L_1} 1$, independent of the word w.

To proceed, observe that by the Kuratowski embedding theorem, we may view K as a subset of ℓ_∞, whose norm we denote by $|\cdot|_\infty$. Fix $r \in (0,1)$ and fix $x, y \in K$. (Eventually, we will choose $r \simeq 1$.) The map f will be a limit of piecewise linear maps $f_n : [0,1] \to \ell_\infty$. In particular, for each $m \in \mathbb{N}$, we will construct a subset $\mathcal{W}_m \subset A^*(r^m)$, a family of nondegenerate closed intervals $\mathscr{E}_m$, and a continuous map $f_m : [0,1] \to \ell_\infty$ satisfying the following properties:

- (P1) The intervals in $\mathscr{E}_m$ have mutually disjoint interiors and their union $\bigcup \mathscr{E}_m = [0,1]$. Furthermore, $f_m(0) = x$ and $f_m(1) = y$.
- (P2) For each $I \in \mathscr{E}_m$, $f_m|I$ is linear and there exists $u \in \mathcal{W}_m$ such that $f_m(\partial I) \subset K_u$ and $|I| \geq L_u^s$. Moreover, if $I, I' \in \mathscr{E}_m$ are distinct, then the corresponding words $u, u' \in \mathcal{W}_m$ are also distinct.
- (P3) For each $I \in \mathscr{E}_{m+1}$, there exists $J \in \mathscr{E}_m$ such that $f_{m+1}|\partial J = f_m|\partial J$. Moreover, $|f_m(p) - f_{m+1}(p)|_\infty < 3r^m$ for all $p \in [0,1]$.

Let us first see how to complete the proof, assuming the existence of family of such maps. On one hand, property (P3) gives

$$\|f_m - f_{m+1}\|_\infty < 3r^m. \tag{3.8}$$

On the other hand, by property (P2), $|I| \geq L_1^s r^{ms}$ and $\mathrm{diam}\, f_m(I) < r^m$ for all $I \in \mathscr{E}_m$. Therefore, for all $p, q \in [0,1]$,

$$|f_m(p) - f_m(q)|_\infty \leq L_1^{-s} r^{m(1-s)} |p - q|. \tag{3.9}$$

By (3.8), (3.9), and Lemma 2.7, the sequence $(f_m)_{m=1}^\infty$ converges uniformly to a $(1/s)$-Hölder map $f : [0,1] \to \ell_\infty$ with $f(0) = x$, $f(1) = y$, and $\mathrm{Höld}_{1/s}\, f \lesssim_{s,L_1,r} 1$. At this stage of the proof, we require that $r \simeq 1$ in order to obtain f with $\mathrm{Höld}_{1/s}(f) \lesssim_{s,L_1} 1$. For example, $r = \frac{1}{2}$ would suffice. Finally, by (P2) and (3.9),

$$\mathrm{dist}(f_m(p), K) \lesssim_{s,L_1} r^m \quad \text{for all } m \in \mathbb{N} \text{ and } p \in [0,1].$$

Therefore, $f([0,1]) \subset K$ and the proposition follows.

It remains to construct $\mathcal{W}_m$, $\mathscr{E}_m$, and f_m satisfying properties (P1), (P2), and (P3). The construction is in an inductive manner.

By Lemma 3.3, there is a set $\mathcal{W}_1 = \{w_1, \dots, w_n\}$ of distinct words in $A^*(r)$, enumerated so that $x \in K_{w_1}$, $y \in K_{w_n}$, and $K_{w_i} \cap K_{w_{i+1}} \neq \emptyset$ for $i \in \{1, \dots, n-1\}$. For each $i \in \{1, \dots, n-1\}$, choose $p_i \in K_{w_i} \cap K_{w_{i+1}}$. To proceed, define $\mathscr{E}_1 = \{I_1, \dots, I_n\}$ to be closed intervals in $[0,1]$ with disjoint interiors, enumerated according to the orientation of $[0,1]$, whose union is $[0,1]$, and such that $|I_j| \geq L_{w_i}^s$ for all $i \in \{1,\dots,n\}$. We are able to find such intervals, since

$$1 = \sum_{u\in A^*(r)} L_u^s \geq \sum_{u\in\mathcal{W}_1} L_u^s.$$

Next, define $f_1 : [0,1] \to \ell_\infty$ in a continuous fashion so that f_1 is linear on each I_i and:

(1) $f_1(0) = x$ and $f_1(I_1)$ is the segment that joins x with p_1;
(2) $f_1(1) = y$ and $f_1(I_n)$ is the segment that joins p_{n-1} with y; and,
(3) for $j \in \{2,\dots,n-1\}$, if any, $f_1(I_j)$ is the segment that joins p_{j-1} with p_j.

Suppose that for some $m \in \mathbb{N}$, we have defined $\mathcal{W}_m \subset A^*(r^m)$, a collection $\mathscr{E}_m$, and a piecewise linear map $f_m : [0,1] \to \ell_\infty$ that satisfy (P1)–(P3). For each $I \in \mathscr{E}_m$, we will define a collection of intervals $\mathscr{E}_{m+1}(I)$ and a collection of words $\mathcal{W}_{m+1}(I) \subset A^*(r^{m+1})$. We then set $\mathscr{E}_{m+1} = \bigcup_{I\in\mathscr{E}_m} \mathscr{E}_{m+1}(I)$ and $\mathcal{W}_{m+1} = \bigcup_{I\in\mathscr{E}_m} \mathcal{W}_{m+1}(I)$. In the process, we will also define f_{m+1}. To proceed, suppose that $I \in \mathscr{E}_m$, say $I = [a,b]$, with I corresponding to the word $w \in \mathcal{W}_m$. Since K is connected, by Lemma 3.3, there exist distinct words $\mathcal{W}_{m+1}(I) = \{w_1,\dots,w_l\} \subset A_w^*(r^{m+1})$ such that $f_m(a) \in K_{w_1}$, $f_m(b) \in K_{w_l}$, and $K_{w_j} \cap K_{w_{j+1}} \neq \emptyset$ for all $j \in \{1,\dots,l-1\}$. Let $\mathcal{E}_{m+1}(I) = \{I_1,\dots,I_l\}$ be closed intervals in I with mutually disjoint interiors, enumerated according to the orientation of I, whose union is I, and such that $a \in I_1$, $b \in I_l$ and $|I_j| \geq L_{w_j}^s$ for all $j \in \{1,\dots,l\}$. We are able to find such intervals, since by our inductive hypothesis,

$$|I| \geq L_w^s = \sum_{u\in A_w^*(r^{m+1})} L_u^s \geq \sum_{i=1}^{l} L_{w_i}^s.$$

For each $j \in \{1,\dots,l-1\}$, choose $p_j \in K_{w_j} \cap K_{w_{j+1}}$.

With the choices above, now define $f_{m+1}|I : I \to \ell_\infty$ in a continuous fashion so that $f_{m+1}|J$ is linear for each $J \in \mathscr{E}_{m+1}(I)$ and:

(1) $f_{m+1}(a) = f_m(a)$ and $f_{m+1}(I_1)$ is the segment that joins y with p_1;
(2) $f_{m+1}(b) = f_m(b)$ and $f_{m+1}(I_l)$ is the segment that joins p_{l-1} with $f_m(b)$; and,
(3) for $j \in \{2,\dots,l-1\}$ (if any), $f_{m+1}(I_j)$ is the segment that joins p_{j-1} with p_j.

Properties (P1), (P2), and the first claim of (P3) are immediate. To verify the second claim of (P3), fix $z \in [0,1]$. By (P1), there exists $I \in \mathscr{E}_{m+1}$ such that $z \in I$. Let J be the unique element of $\mathscr{E}_m$ such that $I \subset J$. Then there exists $w \in A^*(r^m)$ such that $I \in \mathscr{E}_{m+1}(J)$ and $f_m(\partial J) \subset K_w$. Since $f_{m+1}(\partial I) \subset K_u$ for some $u \in A_w^*(r^{m+1})$, we have that $f_{m+1}(\partial I)$. Let $y_1 \in \partial I$ and $y_2 \in \partial J$. We have

$$\begin{aligned}
&|f_m(z) - f_{m+1}(z)|_\infty \\
&\qquad \leq |f_m(z) - f_m(y_2)|_\infty + |f_m(y_2) - f_{m+1}(y_1)|_\infty + |f_{m+1}(y_1) - f_{m+1}(z)|_\infty \\
&\qquad \leq 3\,\mathrm{diam}\, K_w < 3r^m.
\end{aligned}$$ □

3.3. **Hölder parameterization (Proof of Theorem 1.2).** The proof of Theorem 1.2 is modeled after the proof of [BV19, Theorem 2.3], which gave a criterion for the set of leaves of a "tree of sets" in Euclidean space to be contained in a Hölder curve. Here we view the attractor $K_{\mathcal{F}}$ as the set of leaves of a tree, whose edges are Hölder curves.

Proof of Theorem 1.2. Rescaling the metric d, we may assume for the rest of the proof that $\operatorname{diam} K = 1$. Fix $q \in K$, and for each $w \in A^*$, set $q_w := \phi_w(q)$ with the convention $q_\epsilon = q$. Fix $\alpha > s = \text{s-dim}\,\mathcal{F}$ and fix $0 < r < 1$.

Note that for any $m \in \mathbb{N}$ and any $w \in A^*(r^m)$, the set $A^*_w(r^{m+1})$ has at least 1 and at most $L_1^{-s} r^{-s}$ elements. The elements of $A^*_w(r^{m+1})$ are called the *children* of w, and w is called their *parent*. If $u \in A^*_w(r^{m+1})$, then we write $w = p(u)$. Applying Lemma 3.2 twice, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} \sum_{w\in A^*(r^m)} \sum_{u\in A^*_w(r^{m+1})} d(q_w, q_u)^\alpha \\
&\qquad \leq \sum_{m=1}^{\infty} \sum_{w\in A^*(r^m)} \sum_{u\in A^*_w(r^{m+1})} L_w^\alpha \leq L_1^{-s} r^{-s} \sum_{m=1}^{\infty} \sum_{w\in A^*(r^m)} L_w^\alpha \\
&\qquad \leq L_1^{-s} r^{-s} \sum_{m=1}^{\infty} \sum_{w\in A^*(r^m)} r^{\alpha m} \leq L_1^{-2s} r^{-s} \sum_{m=1}^{\infty} r^{(\alpha-s)m} \lesssim_{L_1,s,\alpha,r} 1.
\end{aligned} \tag{3.10}$$

For each $w \in A^*(r^m)$ and $u \in A^*_w(r^{m+1})$, let $f_{w,u} : [0, L_w^s] \to K_w$ be the $(1/s)$-Hölder map with $f_{w,u}(0) = q_w$ and $f_{w,u}(L_w^s) = q_u$ given by Proposition 3.4. Let also $\gamma_{w,u}$ be the image of $f_{w,u}$. We can write K as the closure of the set

$$\Gamma_\circ := \bigcup_{m=1}^{\infty} \bigcup_{w\in A^*(r^{m-1})} \bigcup_{u\in A^*_w(r^m)} \gamma_{w,u}.$$

For each $m \in \mathbb{N}$ and $w \in A^*(r^m)$ define

$$M_w := 2 \sum_{j=m+1}^{\infty} \sum_{u\in A^*_w(r^j)} L_{p(u)}^\alpha \lesssim_{L_1,s,\alpha,r} r^{m\alpha},$$

where the sum over all descendants of w. Setting $M := M_\epsilon$, by (3.10), we have that $M \lesssim_{L_1,s,\alpha,r} 1$. We will construct a $(1/\alpha)$-Hölder continuous surjective map $F : [0, M] \to K$ by defining a sequence $F_m : [0, M] \to K$ whose limit is F and whose image is the truncated tree

$$\Gamma_m := \bigcup_{i=1}^{m} \bigcup_{w\in A^*(r^{i-1})} \bigcup_{u\in A^*_w(r^i)} \gamma_{w,u}.$$

Lemma 3.5. *For each $m \in \mathbb{N}$, there exist two collections $\mathscr{B}_m$, $\mathscr{N}_m$ of nondegenerate closed intervals in $[0,1]$, a bijection $\eta_m : \mathscr{N}_m \to A^*(r^m)$, and a map $F_m : [0, M] \to \Gamma_m$ with the following properties.*

(P1) The families $\mathscr{N}_m$ and $\mathscr{B}_m$ are disjoint, the elements in $\mathscr{N}_m \cup \mathscr{B}_m$ have mutually disjoint interiors, and $\bigcup(\mathscr{N}_m \cup \mathscr{B}_m) = [0, M]$. Moreover, $F_m([0, M]) = \Gamma_m$.

(P2) *If $I \in \mathscr{N}_{m+1}$, then there is $J \in \mathscr{N}_m$ such that $I \subset J$ and $\eta_{m+1}(I) \in A^*_{\eta_m(J)}(r^{m+1})$. Conversely, if $J \in \mathscr{N}_n$, then there exist $J_1 \in \mathscr{N}_{m+1}$ and $J_2 \in \mathscr{B}_{m+1}$ such that $J_1 \subset I$ and $J_2 \subset I$ and* $\text{card}\{I \in \mathscr{B}_{m+1} \cup \mathscr{N}_{m+1} : I \subset J\} \leq L_1^{-s} r^s$.

(P3) *If $I \in \mathscr{B}_{m+1}$, then either $I \in \mathscr{B}_m$ or there exists $J \in \mathscr{N}_m$ such that $I \subset J$. Conversely, $\mathscr{B}_m \subset \mathscr{B}_{m+1}$.*

(P4) *For each $I \in \mathscr{N}_m$, $|I| = M_{\eta_m(I)}$, $F_m|I$ is constant and equal to $q_{\eta(I)}$ and $F_{m+1}|\partial I = F_m|\partial I$.*

(P5) *For each $I \in \mathscr{B}_m$, there exists $w \in A^*(r^{m-1})$ and $u \in A^*_w(r^m)$ such that $|I| = L_w^{\alpha}$ and $F_m|I = f_{w,u} \circ \psi_I$ where ψ_I is (s/α)-Hölder with* $\text{Höld}_{s/\alpha}\,\psi_I = 1$. *Conversely, for any $w \in A^*(r^{m-1})$ and $u \in A^*_w(r^m)$ there exists $I \in \mathscr{B}_m$ as above. Finally, $F_{m+1}|I = F_m|I$ for all $I \in \mathscr{B}_m$.*

We now complete the proof of Theorem 1.2, assuming Lemma 3.5. Let $\mathscr{B}_m$, $\mathscr{N}_m$, η_m and F_m be as in Lemma 3.5. Notice by (P2) that if $I \in \mathscr{N}_m$, then for all $F_n(I) \subset K_{\eta_m(I)}$. We claim that

$$|F_m(x) - F_{m+1}(x)|_\infty \leq 2r^m. \tag{3.11}$$

Equation (3.11) is clear by (P5) if $x \in \mathscr{B}_m$. If $x \in \mathscr{N}_m$, then by (P2) and (P4) there exists $w \in A^*(r^m)$ such that $F_m(I)$ is an element of K_w and $F_m(I) \subset F_{m+1}(I) \subset K_w$. Therefore,

$$|F_m(x) - F_{m+1}(x)|_\infty \leq 2\,\text{diam}\, K_w < 2r^m.$$

We now claim that for all $m \in \mathbb{N}$ and all $x, y \in [0,1]$,

$$|F_m(x) - F_m(y)|_\infty \lesssim_{L_1,s,\alpha,r} r^{m(1-\alpha/s)}|x-y|^{1/s}. \tag{3.12}$$

To prove (3.12) fix $x, y \in [0, M]$ and consider the following cases.

Case 1. Suppose that there exists $I \in \mathscr{B}_m \cup \mathscr{N}_m$ such that $x, y \in I$. If $I \in \mathscr{N}_m$, (3.12) is immediate since $F_m|I$ is constant. If $I \in \mathscr{B}_m$, then by (P5)

$$|F_m(x) - F_m(y)|_\infty \lesssim_{L_1,s,r} \frac{\text{diam}\, f_m(I)}{|I|^{1/s}}|x-y|^{1/s} = r^{m(1-\alpha/s)}|x-y|^{1/s}.$$

Case 2. Suppose that there exist $I_1, I_2 \in \mathscr{B}_m \cup \mathscr{N}_m$ such that $I_1 \cap I_2$ is a single point $\{z\}$, $x \in I_1$ and $y \in I_2$. Then, by triangle inequality and Case 1,

$$|F_m(x) - F_m(y)|_\infty \leq |F_m(x) - F_m(z)|_\infty + |F_m(z) - F_m(y)|_\infty \lesssim_{L_1,s,r} 2r^{m(1-\alpha/s)}|x-y|^{1/s}.$$

Case 3. Suppose that Case 1 and Case 2 do not hold. Let m_0 be the smallest positive integer m such that there exists $I \in \mathscr{B}_m \cup \mathscr{N}_m$ with $x \leq z \leq y$ for all $z \in I$. In particular, suppose that

$$a_1 \leq x \leq a_2 < a_3 < \cdots < a_n \leq y < a_{n+1},$$

where $[a_i, a_{i+1}] \in \mathscr{B}_{m_0} \cup \mathscr{N}_{m_0}$ for all $i \in \{1, \dots, n\}$. By minimality of m_0 and (P2), $n \leq 2L_1^{-s} r^{-s}$. By (P4) and (P5), $|a_i - a_{i+1}| \gtrsim_{L_1,s,\alpha,r} r^{\alpha m_0}$ and $F_m(a_i) = F_{m_0}(a_i)$ for all i. Furthermore, by (P2), (P3) and (P5) we have

$$\max\{|F_m(x) - F_m(a_2)|_\infty, |F_m(y) - F_m(a_n)|_\infty\} \leq r^{m_0}.$$

Therefore, by Case 1 and the triangle inequality,

$$\begin{aligned}
&|F_m(x) - F_m(y)|_\infty \\
&\qquad \leq |F_m(x) - F_m(a_2)|_\infty + \sum_{i=2}^{n-1} |F_m(a_i) - F_m(a_{i+1})|_\infty + |F_m(y) - F_m(a_n)|_\infty \\
&\qquad \lesssim_{L_1,s,r} 2r^{m_0} + r^{m_0(1-\alpha/s)} \sum_{i=2}^{n-1} |a_i - a_{i+1}|^{1/s} \\
&\qquad \lesssim_{L_1,s,r} r^{m_0(1-\alpha/s)} \sum_{i=2}^{n-1} |a_i - a_{i+1}|^{1/s} \\
&\qquad \lesssim_{L_1,s,r} r^{m_0(1-\alpha/s)} \left(\sum_{i=2}^{n-1} |a_i - a_{i+1}| \right)^{1/s} \leq r^{m_0(1-\alpha/s)} |x-y|^{1/s}.
\end{aligned}$$

By (3.11), (3.12) and Lemma 2.7, we have that F_m converges pointwise to a $(1/\alpha)$-Hölder continuous $F : [0, M] \to K$ with $\mathrm{Höld}_{1/\alpha}(F) \lesssim_{L_1,s,\alpha,r} 1$. By (P1), we have that $F([0, 1]) \subset K$ and that $\bigcup_{m\in\mathbb{N}} \Gamma_m \subset F([0, 1])$. Therefore, $F([0, 1]) = K$. This completes the proof of Theorem 1.2, assuming Lemma 3.5. $\square$

Proof of Lemma 3.5. We give the construction of $\mathscr{B}_m$, $\mathscr{N}_m$, η_m and F_m in an inductive manner.

Suppose that $A^*(r) = \{w_1, \dots, w_n\}$. Decompose $[0, M]$ as

$$[0, M] = I_1 \cup J_1 \cup I'_1 \cup \dots \cup I_n \cup J_n \cup I'_n$$

of closed intervals with mutually disjoint interiors, enumerated according to the orientation of $[0, M]$ such that $|I_j| = |I'_j| = 1$ and $|J_j| = M_{w_j}$. Set $\mathscr{B}_1 = \{I_1, I'_1, \dots, I_n, I'_n\}$, $\mathscr{N}_1 = \{J_1, \dots, J_n\}$ and $\eta_1(J_j) = w_j$.

We now define $F_1 : [0, M] \to \Gamma_1$ as follows. For each $J_i \in \mathscr{N}_1$ let $F_1|J_i \equiv q_{w_i}$. For each $i \in \{1, \dots, n\}$, let $\psi_i : I_i \to [0, 1]$ (resp. $\psi'_i : I'_i \to [0, 1]$) be a (s/α)-Hölder orientation preserving (resp. orientation reversing) homeomorphism with $\mathrm{Höld}_{s/\alpha}\, \psi_i = 1$ (resp. $\mathrm{Höld}_{s/\alpha}\, \psi'_i = 1$). Define now $F_1|I_i = f_{\epsilon,w_i} \circ \psi_i$ and $F_1|I'_i = f_{\epsilon,w_i} \circ \psi'_i$. The properties (P1)–(P5) are easy to check.

Suppose now that for some $m \geq 1$, we have constructed $\mathscr{B}_m$, $\mathscr{N}_m$, η_m and F_m satisfying (P1)–(P5). For each $I \in \mathscr{B}_m$ define $F_{m+1}|I = F_m|I$. For each $I \in \mathscr{N}_m$ we construct families $\mathscr{B}_{m+1}(I)$ and $\mathscr{N}_{m+1}(I)$ and then we set

$$\mathscr{B}_{m+1} = \mathscr{B}_m \cup \bigcup_{I\in\mathscr{N}_m} \mathscr{B}_{m+1}(I), \qquad \mathscr{N}_{m+1} = \bigcup_{I\in\mathscr{N}_m} \mathscr{N}_{m+1}(I).$$

In the process we also define F_{m+1} and η_m.

Suppose that $I \in \mathscr{N}_m$ and write $I = [a, b]$. By the inductive hypothesis (P3), there exists $w \in A^*(r^m)$ such that $F_m(I) = q_w$. Suppose that $A^*_w(r) = \{w_1, \dots, w_n\}$. Decompose I as

$$I = I_1 \cup J_1 \cup I'_1 \cup \dots \cup I_l \cup J_l \cup I'_l$$

of closed intervals with mutually disjoint interiors, enumerated according to the orientation of I such that $|I_j| = L_w^\alpha$ and $|J_j| = M_{w_j}$. Set $\mathscr{B}_{m+1}(I) = \{I_1, I_1', \dots, I_l, I_l'\}$, $\mathscr{N}_{m+1}(I) = \{J_1, \dots, J_l\}$ and $\eta_{m+1}|A_w^*(r^{m+1})(J_i) = w_i$.

For each $J_i \in \mathscr{N}_{m+1}(I)$ let $F_{m+1}|J_i \equiv q_{w_i}$. For each $i \in \{1, \dots, l\}$, let $\psi_i : I_i \to [0, L_w^s]$ (resp. $\psi_i' : I_i' \to [0, L_w^s]$) be a (s/α)-Hölder orientation preserving (resp. orientation reversing) homeomorphism with $\mathrm{Höld}_{s/\alpha} \psi_i = 1$ (resp. $\mathrm{Höld}_{s/\alpha} \psi_i' = 1$). Define now $F_{m+1}|I_i = f_{w,w_i} \circ \psi_i$ and $F_1|I_i' = f_{w,w_i} \circ \psi_i'$. The properties (P1)–(P5) are easy to check and are left to the reader. □

4. Hölder parameterization of IFS without branching by arcs

On route to the proof of Theorem 1.3 (see §5), we first parameterize IFS attractors without branching by $(1/s)$-Hölder arcs (see §4.1), where s is the similarity dimension. We then show that under the assumption of bounded turning, self-similar sets without branching are $(1/s)$-bi-Hölder arcs (see §4.2).

4.1. **IFS without branching.** Given an IFS $\mathcal{F} = \{\phi_i : i \in A\}$ over a complete metric space, we say that $\mathcal{F}$ has *no branching* or is *without branching* if for every $m \in \mathbb{N}$ and word $w \in A^m$ (see §3.1), there exist at most two words $u \in A^m \setminus \{w\}$ such that $\phi_w(K_{\mathcal{F}}) \cap \phi_u(K_{\mathcal{F}}) \neq \emptyset$.

Proposition 4.1 (parameterization of connected IFS without branching)**.** *Let $\mathcal{F}$ be an IFS over a complete metric space; let $s = \text{s-dim}(\mathcal{F})$. If $K_{\mathcal{F}}$ is connected,* $\operatorname{diam} K_{\mathcal{F}} > 0$*, and $\mathcal{F}$ has no branching, then there exists a $(1/s)$-Hölder homeomorphism $f : [0,1] \to K$ with* $\mathrm{Höld}_{1/s} f \lesssim_{L_1,s} \operatorname{diam} K$*, where $L_1 = \min_{\phi\in\mathcal{F}} \operatorname{Lip} \phi$.*

For the rest of §4.1, fix an IFS $\mathcal{F} = \{\phi_1, \dots, \phi_k\}$ over a complete metric space (X, d) whose attractor $K := K_{\mathcal{F}}$ is connected and has positive diameter. Adopt the notation and conventions set in the first paragraph of §3 as well as in §3.1. In addition, assume that $\mathcal{F}$ has no branching. Since $\operatorname{diam} K > 0$, $k \geq 2$. Replacing $\mathcal{F}$ with the iterated IFS $\mathcal{F}' = \{\phi_w : w \in A^2\}$ if needed, we may assume without loss generality that $k \geq 4$. Finally, rescaling the metric d, we may assume without loss of generality that $\operatorname{diam} K = 1$ (see the proof of Proposition 3.4).

Given $n \in \mathbb{N}$ and $w \in A^n$, we define the *valence* of w in A^n by

$$\mathrm{val}(w, A^n) := \operatorname{card}\{u \in A^n \setminus \{w\} : K_u \cap K_w \neq \emptyset\}.$$

Lemma 4.2. *For each $n \in \mathbb{N}$, there exist exactly two distinct words $w \in A^n$ such that* $\mathrm{val}(w, A^n) = 1$*; for all other $u \in A^n$, we have* $\mathrm{val}(u, A^n) = 2$*.*

Proof. By the no branching property, we have that $\mathrm{val}(w, A^n) \in \{1, 2\}$ for all $n \in \mathbb{N}$ and $w \in A^n$. To finish the proof, it suffices to show that, for each $n \in \mathbb{N}$, there exists at least one $w \in A^n$ such that $\mathrm{val}(w, A^n) = 1$. We apply induction on n.

Suppose $n = 1$ and, for a contradiction, assume that for all $i \in A$, $\mathrm{val}(i, A) = 2$. Fix $i \in A \setminus \{1\}$ such that $K_i \cap K_1 \neq \emptyset$. There exist $j, j_1 \in A$ such that $K_{1j_1} \cap K_{ij} \neq \emptyset$.

By our assumption, there exist distinct $j_2, j_3 \in A \setminus \{j_1\}$ such that $K_{j_1} \cap K_{j_2} \neq \emptyset$ and $K_{j_1} \cap K_{j_3} \neq \emptyset$. Therefore, $K_{1j_1} \cap K_{1j_2} \neq \emptyset$ and $K_{1j_1} \cap K_{1j_3} \neq \emptyset$. But then $\mathrm{val}(1j_1, A^2) \geq 3$ which is false.

Assume now the lemma to be true for some n. Let $w \in A^n$ with $\mathrm{val}(w, A^n) = 1$ and let i_0, j_0 be the unique elements $i \in A$ such that $\mathrm{val}(i, A) = 1$. We claim that one of $\mathrm{val}(wi_0, A^{n+1})$, $\mathrm{val}(wj_0, A^{n+1})$ is equal to 1. Let u be the unique element of $A^n \setminus \{w_0\}$ such that $K_w \cap K_u \neq \emptyset$. It suffices to show that one of $K_{wi_0} \cap K_u$, $K_{wj_0} \cap K_u$ is empty. For a contradiction, assume that both sets are nonempty.

Let $i, j \in A$ such that $K_{wi_0} \cap K_{ui} \neq \emptyset$ and $K_{wj_0} \cap K_{uj} \neq \emptyset$. We claim that $\{i, j\} = \{i_0, j_0\}$. To prove the claim, assume first that $i \notin \{i_0, j_0\}$. Then there exist distinct $i_1, i_2 \in A \setminus \{i\}$ such that $K_{ui} \cap K_{ui_l} \neq \emptyset$ for $l = 1, 2$ and $\mathrm{val}(ui, A^{n+1}) \geq 3$ which is false. So, $\{i, j\} \subseteq \{i_0, j_0\}$. If $i = j$, then there exists $i' \in A$ such that $K_{ui} \cap K_{ui'} \neq \emptyset$ and $\mathrm{val}(ui, A^{n+1}) \geq 3$ which is false. So, $i \neq j$ and $\{i, j\} = \{i_0, j_0\}$.

Notice that $\mathrm{val}(u, A^n) = 2$ as, otherwise, $K_w \cup K_u$ would be a component of K and K would be disconnected. Let $u' \in A^n \setminus \{w\}$ such that $K_u \cap K_{u'} \neq \emptyset$. Let $p \in A$ with $K_{up} \cap K_{u'} \neq \emptyset$. If $p \in \{i_0, j_0\}$, then $\mathrm{val}(up, A^{n+1}) \geq 3$ because K_{up} intersects one of K_{wi_0}, K_{wj_0}, a set K_{ul} for some $l \in A \setminus \{p\}$ and a set $K_{u'q}$ for some $q \in A$. If $p \notin \{i_0, j_0\}$, then $\mathrm{val}(up, A^{n+1}) \geq 3$ because K_{up} intersects two sets K_{ul_1}, K_{ul_2} for some distinct $l_1, l_2 \in A \setminus \{p\}$ and a set $K_{u'q}$ for some $q \in A$. In either case, we arrive to a contradiction. $\square$

From Lemma 4.2, we obtain two simple corollaries.

Lemma 4.3. *For all $n \in \mathbb{N}$ and all $w, u \in A^n$, $K_w \cap K_u$ is at most a point.*

Proof. Fix $w, u \in A^n$ such that $K_w \cap K_u \neq \emptyset$. We first claim that there exists unique $i \in A$ and unique $j \in A$ such that $K_{wi} \cap K_{uj} \neq \emptyset$. Assuming the claim to be true, we have

$$\mathrm{diam}(K_w \cap K_u) = \mathrm{diam}(K_{wi} \cap K_{uj}) \leq L_k \,\mathrm{diam}(K_w \cap K_u) < \mathrm{diam}(K_w \cap K_u)$$

which implies that $\mathrm{diam}(K_w \cap K_u) = 0$.

To prove the claim, fix $i \in A$ such that $K_{wi} \cap K_u \neq \emptyset$. Following the arguments in the proof of Lemma 4.2, we have that $i \in \{i_0, j_0\}$ where $\{i_0, j_0\}$ are the unique elements of A with valence 1 in A; say $i = i_0$. If there exists $w' \in A \setminus \{w, u\}$ such that $K_{w'} \cap K_w \neq \emptyset$, then by Lemma 4.2 $K_{w'} \cap K_{wj_0} \neq \emptyset$ and $K_{wj_0} \cap K_u = \emptyset$. If no such w' exists, then $\mathrm{val}(w, A^n) = 1$ which implies that $\mathrm{val}(wj_0, A^{n+1}) = 1$ which also implies $K_{wj_0} \cap K_u = \emptyset$. In either case, $K_{wj_0} \cap K_u = \emptyset$ and i is unique. $\square$

Lemma 4.4. *For all $n \in \mathbb{N}$, there exist exactly two words $w \in A^n$ such that the set $K_w \cap \overline{K \setminus K_w}$ contains only one point.*

Proof. By Lemma 4.2, for each $n \in \mathbb{N}$, there exist exactly two distinct words $w, u \in A^n$ such that $\mathrm{val}(w, A^n) = \mathrm{val}(u, A^n) = 1$. Fix such a word, say w. There exists unique $w' \in A^n \setminus \{w\}$ such that $K_w \cap K_{w'} = K_w \cap \overline{K \setminus K_w}$. By Lemma 4.3, the latter intersection is a single point. $\square$

We are ready to prove Proposition 4.1.

Proof of Proposition 4.1. By Lemma 4.4, there exist two infinite words $w_0, w_1 \in A^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $w_0(n)$ and $w_1(n)$ are the unique words in $w \in A^n$ such that $\mathrm{val}(w, A^n) = 1$. Set

$$\{v_0\} = \bigcap_{n=1}^{\infty} K_{w_0(n)} \qquad \text{and} \qquad \{v_1\} = \bigcap_{n=1}^{\infty} K_{w_1(n)}.$$

Fix $r \in (0,1)$ and let $f : [0,1] \to K$ be the map given by Proposition 3.4 with $x = v_0$ and $y = v_1$. We already have that $f([0,1]) \subset K$. We claim that for all $m \in \mathbb{N}$ and all $w \in A^*(r^m)$, we have $f([0,1]) \cap K_w \neq \emptyset$. Assuming the claim, it follows that $\mathrm{dist}(x, f([0,1])) \leq r^m$ for all $x \in K$ and all $m \in \mathbb{N}$. Hence $K \subset f([0,1])$ and $K = f([0,1])$.

Let $N = \max\{n \in \mathbb{N} : A^*(r^m) \cap A^n \neq \emptyset\}$. To prove the claim fix $w \in A^*(r^m)$. By Lemma 3.1, there exists $u \in A^N$ such that $K_u \subseteq K_w$. If $u \in \{w_0(N), w_1(N)\}$, then K_w contains one of v_0, v_1, so $f([0,1]) \cap K_w \neq \emptyset$. If $u \notin \{w_0(N), w_1(N)\}$, then $\mathrm{val}(u, A^N) = 2$ and by Lemma 4.2, $K \backslash K_u$ has two components, one containing v_0 and the other containing v_1. Since $f([0,1])$ is connected and contains v_0, v_1, $\emptyset \neq f([0,1]) \cap K_u \subseteq f([0,1]) \cap K_w$.

It remains to show that f is a homeomorphism and suffices to show that f is injective. Recall the definitions of $\mathscr{E}_m$ and f_m from the proof of Proposition 3.4. By (P2) and (P3) therein, for each $m \in \mathbb{N}$ and $I \in \mathscr{E}_m$, there exists $w_I \in A^*(r^m)$ such that $f(I) \subset K_{w_I}$. Moreover, $w_I \neq w_J$ if $I \neq J$. In conjunction with the fact that $f([0,1]) = K$, we have that $f(I) = K_{w_I}$. By design of the map f, it is easy to see that $K_{w_I} \cap K_{w_J}$ if and only if $I \cap J$. Assume $x, y \in [0,1]$ with $x \neq y$. Then there exists $m \in \mathbb{N}$ and disjoint $I, J \in \mathscr{E}_m$ such that $x \in I$ and $y \in J$. Hence $K_{w_I} \cap K_{w_J} = \emptyset$. Therefore, $f(I) \cap f(J) = \emptyset$, which yields $f(x) \neq f(y)$. □

From the proof of Proposition 4.1, for each $m \in \mathbb{N}$, there is a one-to-one correspondence between intervals I in $\mathscr{E}_m$ and words $w_I \in A^*(r^m)$ with the rule $f(I) = K_{w_I}$.

Corollary 4.5. *For all $m \in \mathbb{N}$ and all $I \in \mathscr{E}_m$, we have $|I| = L_{w_I}^s \simeq r^{ms}$.*

Proof. It suffices to establish the first equality. The proof is by induction on m. For $m = 0$ it is clear since $\mathscr{E}_0 = \{[0,1]\}$ and $L_\epsilon = 1$. Assume the claim to be true for some $m \geq 0$.

Fix $I \in \mathscr{E}_m$ and recall the definition of $\mathscr{E}_{m+1}(I)$ from the proof of Proposition 3.4. Then

$$\{w_J : J \in \mathscr{E}_{m+1}(I)\} = A_{w_I}^*(r^{m+1}).$$

Therefore, by (P2) in the proof of Proposition 3.4 and following the arguments in the proof of Lemma 3.2,

$$L_{w_I}^s = |I| = \sum_{J \in \mathscr{E}_{m+1}(I)} |J| \geq \sum_{u \in A_{w_I}^*(r^{m+1})} L_u^s = L_{w_I}^s.$$

The above can be true if and only if $|J| = L_{w_J}^s$ for all $J \in \mathscr{E}_{m+1}(I)$. As $\mathscr{E}_{m+1} = \bigcup_{I \in \mathscr{E}_m} \mathscr{E}_{m+1}(I)$, we obtain the inductive step and the proof follows. □

4.2. **Bounded turning and self-similar bi-Hölder arcs.** With additional information on the contractions of $\mathcal{F}$ and how the components $K_i = \phi_i(K)$ of the attractor K intersect, the map f constructed in Proposition 4.1 is actually a $(1/s)$-bi-Hölder homeomorphism. We say that K has *bounded turning* if there exists $C \geq 1$ such that for all distinct $i, j \in A$ with $K_i \cap K_j \neq \emptyset$: if $x \in K_i$, $y \in K_j$. and $z \in K_i \cap K_j$, then

$$d(x,y) \geq C^{-1} \max\{d(x,z), d(y,z)\}. \tag{4.1}$$

In general, self-similar curves (even in $\mathbb{R}^2$) do not have the bounded turning property; see [ATK03, Example 2.3] by Aseev, Tetenov, and Kravchenko.

Proposition 4.6 (self-similar sets without branching and with bounded turning)**.** *Let $\mathcal{F}$ be an IFS over a complete metric space that is generated by similarities; let $s = \text{s-dim}(\mathcal{F})$. If $K_{\mathcal{F}}$ is connected,* $\operatorname{diam} K_{\mathcal{F}} > 0$*, $\mathcal{F}$ has no branching, and $K_{\mathcal{F}}$ is bounded turning, then there exists a $(1/s)$-bi-Hölder homeomorphism $f : [0,1] \to K$.*

Proof. Fix $x, y \in [0,1]$ with $x < y$. Let $m_0 \in \mathbb{N}$ be the smallest integer m such that there exists $I \in \mathscr{E}_m$ such that $I \subset [x,y]$. Fix now $I \in \mathscr{E}_{m_0}$ as above. The proof is divided into two cases.

Case 1. Suppose that there exists $J \in \mathscr{E}_{m_0-1}$ such that $I \subset J$ and $[x,y] \subset J$. Then there exist $w \in A^*(r^{m_0-1})$ and distinct $u_1, u_2 \in A^*$ such that $f(J) = K_w$, $f(x) \in K_{wi}$, $f(y) \in K_{wj}$ and $K_{wi} \cap K_{wj} = \emptyset$. By Corollary 4.5,

$$d(f(x), f(y)) \gtrsim \operatorname{diam} K_w \simeq (\operatorname{diam} J)^{1/s} \geq |x-y|^{1/s}.$$

Case 2. Suppose that Case 1 does not hold. Then there exist distinct $J_1, J_2 \in \mathscr{E}_{m_0-1}$ such that $J_1 \cap [x,y] \neq \emptyset$, $J_2 \cap [x,y] \neq \emptyset$, $[x,y] \subset J_1 \cup J_2$, $I \subset J_1$ and $J_1 \cap J_2$ is a point. Suppose, moreover, that $f(J_1) = K_{w_1}$ and $f(J_2) = K_{w_2}$ with $w_1, w_2 \in A^*(r^{m_0-1})$.

Let w_0 be the longest word such that $K_{w_1} \cup K_{w_2} \subset K_{w_0}$. Then there exist $i_1, i_2 \in A$ such that $K_{w_1} \subset K_{w_0 i_1}$ and $K_{w_2} \subset K_{w_0 i_2}$. Therefore, if z is the unique point of $J_1 \cap J_2$, then by (4.1) and the fact that ϕ_{w_0} is a similarity,

$$d(f(x), f(y)) = L_{w_0} d(\phi_{w_0}^{-1}(f(x)), \phi_{w_0}^{-1}(f(y))) \geq C^{-1} d(f(x), f(z)) + d(f(y), f(z)).$$

Now we have that $I \subset [x,z] \subset J$ so we can apply Case 1 for x, z and use the maximality of I to get,

$$d(f(x), f(y)) \geq C^{-1} d(f(x), f(z)) \gtrsim_C |x-y|^{1/s} \geq |I|^{1/s} \gtrsim_{L_1,s} |x-y|^{1/s}. \qquad \square$$

5. Hölder parameterization of self-similar sets (Remes' method)

Our goal in this section is to record a proof of Theorem 1.3 that combines original ideas of Remes [Rem98] with our style of Hölder parameterization from above. Thus, fix an IFS $\mathcal{F} = \{\phi_1, \dots, \phi_k\}$ over a complete metric space (X,d); let $s = \text{s-dim}(\mathcal{F})$. Assume that $\mathcal{F}$ is generated by similarities, $K_{\mathcal{F}}$ is connected, $\operatorname{diam} K_{\mathcal{F}} > 0$, and $\mathcal{H}^s(K_{\mathcal{F}}) > 0$, where $s = \text{s-dim}(\mathcal{F})$. Recall that $\mathcal{H}^s(K_{\mathcal{F}}) > 0$ implies $\mathcal{F}$ satisfies the strong open set condition by Theorem 2.2. Moreover, by Lemma 2.4, $K_{\mathcal{F}}$ is Ahlfors s-regular; thus, we can find constants $0 < C_1 \le C_2 < \infty$ such that

$$C_1\rho^s \le \mathcal{H}^s(K_{\mathcal{F}} \cap B(x,\rho)) \le C_2\rho^s \quad \text{for all } x \in K \text{ and all } 0 < \rho \le 1. \tag{5.1}$$

As usual, we adopt the notation and conventions set in the first paragraph of §3 as well as in §3.1. If $\mathcal{F}$ has no branching (see §4.1), then a $(1/s)$-Hölder parameterization of $K_{\mathcal{F}}$ already exists by Proposition 4.1. Thus, we shall assume $\mathcal{F}$ has *branching*, i.e. there exists $m \in \mathbb{N}$ and distinct words $w_1, \dots, w_4 \in A^m$ such that $K_{w_1} \cap K_{w_i} \neq \emptyset$ for each $i \in \{2,3,4\}$. In the event that $m \ge 2$ (see Example 5.1), we replace $\mathcal{F}$ with the self-similar IFS $\mathcal{F}' = \{\phi_w : w \in A^m\}$. This causes no harm to the proof, because the attractors coincide, i.e. $K_{\mathcal{F}'} = K_{\mathcal{F}}$, and s-dim$(\mathcal{F}')$ = s-dim$(\mathcal{F})$. Therefore, without loss of generality, we may assume that there exist distinct letters $i_1, i_2, i_3, i_4 \in A$ such that

$$K_{i_1} \cap K_{i_j} \neq \emptyset \quad \text{for each } j \in \{2,3,4\}. \tag{5.2}$$

Example 5.1. Divide the unit square into 3×3 congruent subsquares with disjoint interiors S_i $(1 \le i \le 9)$. Let S_9 denote the central square and for each $1 \le i \le 8$, let $\psi_i : \mathbb{R}^2 \to \mathbb{R}^2$ be the unique rotation-free and reflection-free similarity that maps $[0,1]^2$ onto S_i. The attractor of the IFS $\mathcal{G} = \{\psi_1, \dots, \psi_8\}$ is the *Sierpiński carpet.* Looking only at the intersection pattern of the first iterates $\psi_1(K_{\mathcal{G}})$,…,$\psi_8(K_{\mathcal{G}})$, it appears that $\mathcal{G}$ is without branching. However, upon examining the intersections of the second iterates $\psi_i \circ \psi_j(K_{\mathcal{G}})$ $(1 \le i, j \le 8)$, it becomes apparent that $\mathcal{G}$ has branching.

To continue, use the Kuratowski embedding theorem to embed (K, d) into $(\ell_\infty, |\cdot|_\infty)$. Let d_H denote the Hausdorff distance between compact sets in ℓ_∞. By the Arzelá-Ascoli theorem, to complete the proof of Theorem 1.3, it suffices to establish the following claim.

Proposition 5.2. *There exists a sequence $(F_N)_{N=1}^\infty$ of $(1/s)$-Hölder continuous maps $F_N : [0,1] \to \ell_\infty$ with uniformly bounded Hölder constants such that*

$$\lim_{N\to\infty} d_H(F_N([0,1]), K) = 0.$$

Remark 5.3. It is perhaps unfortunate that we have to invoke the Arzelá-Ascoli theorem to implement Remes' method. We leave as an *open problem* to find a proof of Theorem 1.3 that avoids taking a subsequential limit of the intermediate maps; cf. the proofs in §3 above or the proof of the Hölder traveling salesman theorem in [BNV19].

We devote the remainder of this section to proving Proposition 5.2. To start, fix an index $N \in \mathbb{N}$.

5.1. **Nets.** By the strong open set condition, there exists an open set $U \subset X$ such that $U \cap K \neq \emptyset$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for all $i, j \in A$ with $i \neq j$. Fix $v \in U \cap K$, choose $\tau \in (0, 1/2)$ such that $B_X(v, \tau) \subset U$, and set $r := \frac{1}{4}L_1\tau$.

For each $m \in \{1, \dots, N\}$, put $Y_m := \{\phi_w(v) : w \in A^*(r^m)\}$ (to recall the definition of $A^*(\delta)$, see §3.1). We also define a nested sequence of sets $(V_m)_1^N$ recursively, as follows. Set $V_N := Y_N$. Next, assume we have defined $V_m, \dots, V_N$ for some $2 \le m \le n$ so that

(1) $V_m \subset V_{m+1} \subset \cdots \subset V_N = Y_N$; and,
(2) for each $i \in \{m, \dots, N\}$ and each $w \in A^*(r^i)$, there exists a unique $x \in K_w \cap V_i$.

Replace each $x \in Y_{m-1}$ by an element $x' \in V_m \cap K_{u_x}$ of shortest distance to x, where $u_x \in A^*(r^{m-1})$ is such that $\phi_{u_x}(v) = x$. This produces the set V_{m-1}.

Lemma 5.4. *Let $m \in \{1, \dots, N\}$.*

(1) If $m \leq N-1$, then $V_m \subset V_{m+1}$.
(2) For each $w \in A^*(r^m)$, there exists unique $x \in V_m \cap K_w$.
(3) If $w \in A^*(r^m)$ and $x \in V_m \cap K_w$, then $|x - \phi_w(v)|_\infty \leq 2r^{m+1} = (\frac{1}{2}\tau L_1)r^m$.
(4) For all distinct $a, b \in V_m$ we have $|a-b|_\infty \geq (L_1\tau)r^m$.

Proof. The first two claims follow immediately by design of the sets (V_m).

For the third one we apply backward induction on m. If $m = N$, then by definition of V_N, for each $w \in A^*(r^N)$ and every $x \in V_N \cap K_w$ we have $x = \phi_w(v)$. Assume that the claim is true for $m+1, \dots, N$ and let $w \in A^*(r^m)$ and $x \in V_m \cap K_w$. There exists $wu \in A^*(r^{m+1})$ such that $x \in V_{m+1} \cap K_{wu}$. On one hand, $|\phi_{wu}(v) - \phi_w(v)|_\infty \leq r^{m+1}$ and on the other hand, by induction, $|x - \phi_{wu}(v)|_\infty \leq 2r^{m+2}$. Since $r = \frac{1}{4}L_1\tau$,

$$|x - \phi_w(v)|_\infty \leq r^{m+1} + 2r^{m+2} \leq 2r^{m+1} = (\tfrac{1}{2}\tau L_1)r^m.$$

Finally, for the last claim, if $a, b \in V_m$ are distinct with $a \in K_w \cap V_m$ and $b \in K_u \cap V_m$ and $w, u \in A^*(r^m)$ are distinct, then

$$\begin{aligned}|a-b|_\infty &\geq |\phi_w(v) - \phi_u(v)|_\infty - |a - \phi_w(v)|_\infty - |b - \phi_u(v)|_\infty \\ &\geq r^m - 2(2r^{m+1}) \geq 2L_1\tau r^m - 2(2r^{m+1}) = (L_1\tau)r^m.\end{aligned}$$

□

5.2. **Trees.** We define a finite sequence of trees $T_m = (V_m, E_m)_{m=1,\dots,N}$ in an inductive manner. Note first that for each $m \in \{1, \dots, N\}$ and any $x \in V_m$, there exists unique $w \in A^*(r^m)$ such that $x \in K_w$; we denote this word w by $x(m)$.

Let $G_1 = (V_1, \hat{E}_1)$ where

$$\hat{E}_1 = \{\{x,y\} : x \neq y \text{ and } K_{x(1)} \cap K_{y(1)} \neq \emptyset\}.$$

The connectedness of K implies that G_1 is a connected graph, but not necessarily a tree. Now, removing some edges from $\hat{E}_1$ we obtain a new set E_1 so that $T_1 = (V_1, E_1)$ is a connected tree. Because we assumed $\mathcal{F}$ has branching (see (5.2)), we may assume that T_1 has at least one branch, i.e., there exists a point $x \in V_m$ with valence in T_m at least 3.

Suppose that we have defined $T_m = (V_m, E_m)$ for some $m \in \{1, \dots, N-1\}$. For each $x \in V_m$, let $V_{m+1,x} = V_{m+1} \cap K_{x(m)}$ and let $T_{m+1,x} = (V_{m+1,x}, E_{m+1,x})$ be a connected tree such that $\{y,z\} \in E_{m+1,x}$ only if $y, z \in V_{m+1,x}$, $y \neq z$ and $K_{y(m+1)} \cap K_{z(m+1)} \neq \emptyset$. Moreover, since $K_{x(m)}$ is congruent to K, we may assume that $T_{m+1,x}$ has at least one branch. Now, if $\{a,b\} \in E_m$, there exists $a' \in V_{m+1,a}$ and $b' \in V_{m+1,b}$ such that $K_{a'(m+1)} \cap K_{b'(m+1)} \neq \emptyset$. Set

$$E_{m+1} := \bigcup_{x \in V_m} E_{m+1,x} \cup \bigcup_{\{a,b\} \in E_m} \{\{a', b'\}\}.$$

Below, all trees T_m are realized in ℓ_∞ through the natural identification of $\{v,u\} \in E_m$ with the line segment $[v,u]$.

We now recall a key lemma from Remes [Rem98], which will allow us to find a good way to parameterize the final tree T_N. In the statement of the lemma and in §5.3, given $a,b \in V_N$ with $a \neq b$, $R_N(a,b)$ denotes the unique arc (the "road") in T_N with endpoints a and b.

Lemma 5.5 ([Rem98, Lemma 4.11])**.** *Let $a,b \in V_N$ with $a \neq b$ and let $R \subset V_N$ be the set of vertices of $R_N(a,b)$. Suppose that there exists $m \leq N$ such that $|a-b|_\infty \geq (\frac{1}{2}L_1\tau)r^m$ and $|a-x|_\infty \leq (4/r)r^m$ for all $x \in R$.*

1. *There exists $C \geq 1$ depending only on L_1, s, τ, C_1, C_2 such that the number of the branches of T_N with respect to the arc $R_N(a,b)$ that contain points in $V_m \setminus R$, is less than C.*
2. *There exists $C' \geq 1$ depending only on L_1, s, τ, C_1, C_2 such that if $z_1, \dots, z_l$ are consecutive vertices in $R_N(a,b)$ with $|z_i - z_{i+1}|_\infty \geq (\frac{1}{2}L_1\tau)r^m$, then $l \leq C'$.*
3. *There is $t \in \mathbb{N}$ depending only on L_1, s, τ, C_1, C_2 such that if $m \leq N-t$, then the number of branches of T_N with respect to $R_N(a,b)$ that contain some vertex in $V_{t+m} \setminus R$ is at least $2C + C' + 2$. Moreover, if $c \in V_{t+m}$ is such a vertex, $c \in K_w$ for some $w \in A^*(r^{t+m})$, and if $c' \in K_w$, then c and c' belong to the same branch of $R_N(a,b)$.*

Although [Rem98, Lemma 4.11] is stated and proved for self-similar sets in Euclidean space that satisfy the open set condition, the essential fact in Remes' proof of this lemma is that the attractor K is Ahlfors s-regular, i.e. K satisfies (5.1). Rather than repeat the proof here, we refer the reader to [Rem98] for details. This is the only part in the proof of Theorem 1.3 that the assumption $\mathcal{H}^s(K) > 0$ is required.

5.3. **Two preliminary parameterizations.** Let t be as in Lemma 5.5. For each $m \leq N$, we denote by $T_{N,m}$ the minimal subgraph of T_N that contains V_m. Clearly, $T_N = T_{N,N}$ and $T_{N,m} \subset T_{N,n}$ when $m \leq n$.

For each $m \leq N$ we construct two sets $\mathscr{E}_m$, $\mathscr{B}_m$ of open intervals in $[0,1]$, two sets $\mathscr{P}_m$, $\mathscr{F}_m$ of closed nondegenerate intervals in $[0,1]$ and two continuous maps $f_m : [0,1] \to \ell_\infty$ and $g_m : [0,1] \to T_N$ with the following properties.

- (P1) The families $\mathscr{P}_m, \mathscr{E}_m$ are disjoint and the elements in $\mathscr{P}_m \cup \mathscr{E}_m$ are mutually disjoint. The families $\mathscr{F}_m, \mathscr{B}_m$ are disjoint and the elements in $\mathscr{F}_m \cup \mathscr{B}_m$ are mutually disjoint. Moreover, $[0,1] = \bigcup(\mathscr{P}_m \cup \mathscr{E}_m) = \bigcup(\mathscr{F}_m \cup \mathscr{B}_m)$.
- (P2) For each $I \in \mathscr{E}_m$, $f_m|I$ is linear and for each $I \in \mathscr{P}_m$, $f_m|I$ is constant with value in V_N.
- (P3) For each $I \in \mathscr{B}_m$, $g_m|I$ is linear onto an edge of T_N and for each $I \in \mathscr{F}_m$, $g_m|I$ is constant with value in V_N. Moreover, g_m traces every edge of $T_{N,m}$ exactly twice, once in each direction.
- (P4) We have $\mathscr{P}_m \subset \mathscr{F}_m$ and for each $I \in \mathscr{P}_m$, $f_m|I = g_m|I$.

(P5) If $I = (a,b) \in \mathscr{E}_m$, then $|g_m(a) - g_m(b)|_\infty \geq (\frac{1}{2}L_1\tau)r^m$ and for all $x \in I$, we have $|g_m(a) - g_m(x)|_\infty \leq (5/r)r^m$.

(P6) If $m \leq N-t$ and $I \in \mathscr{E}_m$, then there exists $w \in A^*(r^{m+t})$ such that $g_m(I)$ traverses the vertices of $K_w \cap V_N$.

5.3.1. *Step 1.* Define a continuous map $h_1 : [0,1] \to T_{N,1}$ so that it traverses every edge of $T_{N,1}$ linearly exactly twice, once in each direction. We assume that $h_1(0), h_1(1) \in V_1$ and that every component of $h_1^{-1}(V_N)$ is nondegenerate. Note that $h(0) = h(1)$.

Let $\mathscr{P}_1$ be the collection of components of $h_1^{-1}(V_1)$, let $\mathscr{E}_1$ be the set of components of $[0,1] \setminus \bigcup \mathscr{P}_1$, and let $\mathscr{F}_1'$ be the collection of components of $h_1^{-1}(V_N)$. Let $f_1 : [0,1] \to \ell_\infty$ be a continuous map such that $f_1|I = h_1|I$ if $I \in \mathscr{P}_1$ and $f_1|I$ is linear if $I \in \mathscr{E}_1$.

We now modify h_1 on each interval in $I \in \mathscr{E}_1$ which gives the definition of g_1. Let $\{I_1, \dots, I_{2n_1}\}$ be an enumeration of $\mathscr{E}_1$. Write $I_1 = (x,y)$ and let $a = h_1(x)$ and $b = h_1(y)$. By Lemma 5.4(4), $|a-b|_\infty \geq L_1\tau r > (\frac{1}{2}L_1\tau)r$ and $|a-z|_\infty < (1/r)r$ for every vertex z of $R_N(a,b)$. By Lemma 5.5, there exist at least $2C+2$ branches of $T_{N,t+1}$ with respect to the road $R_N(a,b)$ such that for every branch there exists $w \in A^*(r^{t+1})$ such that all vertices of K_w are in that branch. Fix such a branch B and let $J \in \mathscr{F}_1'$ be a closed nondegenerate interval for which the point $h_1(J)$ belongs to both $h_1(I_1)$ and B. If l is the number of edges of B, subdivide J into $2l$ open and $2l+1$ closed nondegenerate intervals. Define $g_1|I_1 \setminus J = h_1|I_1 \setminus J$ and define $g_1|J$ continuously so that

1. $g_1|J$ traces every edge of B linearly exactly twice, once in each direction,
2. $g_1|\partial J = h_1|\partial J$,
3. for every vertex $z \in V_N \cap B$, every component of $g_1^{-1}(z)$ is a nondegenerate closed interval.

Suppose that we have defined g_1 on $I_1, \dots, I_i$. For I_{i+1} we work exactly as with I_1, but we choose a branch B that has not been traced by $g_1|I_1 \cup \dots \cup I_i$. We can do so because by Lemma 5.5, at most $2C+1$ branches could have been traced by previous choices. Modifying h_1 on each I_i completes the definition of g_1.

Define now $\mathscr{F}_1$ to be the collection of components of $g_1^{-1}(V_N)$ and $\mathscr{B}_1$ be the collection of components of $[0,1] \setminus g_1^{-1}(V_N)$.

Properties (P1)–(P4) follow immediately. The first assertion of (P5) follows from Lemma 5.4(4) and the fact that, for each $I \in \mathscr{E}_1$, $g_1(\partial I) \subset V_1$ The second assertion of (P5) is trivial, as $\operatorname{diam} K = 1$. Property (P6) follows from Lemma 5.5(3).

5.3.2. *Inductive step for $m+1 \leq N-t$.* Suppose that for some $m \leq N-T-1$ we have defined families of intervals $\mathscr{E}_m$, $\mathscr{P}_m$, $\mathscr{B}_m$, $\mathscr{F}_m$, a continuous $g_m : [0,1] \to T_N$ whose image contains $T_{N,m}$ and some branches of $T_{N,m+t}$, and a continuous $f_m : [0,1] \to \ell_\infty$ so that properties (P1)–(P6) hold.

We start by defining an auxiliary map $h_{m+1} : [0,1] \to T_N$. Let $G_1, \dots, G_l$ be the connected components of $\overline{T_{N,m+1} \setminus g_m([0,1])}$. For each $i \in \{1, \dots, l\}$ let x_i be the unique point of $G_i \cap g_m([0,1])$ and let $J_i \in \mathscr{F}_m$ be such that $g_m(J_i) = x_i$. For each I in the set $\mathscr{B}_m \cup (\mathscr{F}_m \setminus \{J_1, \dots, J_l\})$, define $h_{m+1}|I = g_m|I$. Fix now $i \in \{1, \dots, l\}$ and define $h_{m+1}|J_i : J_i \to G_i$ continuously so that

(1) $h_{m+1}|\partial J_i = g_m|\partial J_i$
(2) $h_{m+1}|J_i$ traverses every edge of G_i linearly exactly twice, once in each direction,
(3) every component of $(h_{m+1}|J_i)^{-1}(V_N)$ is a nondegenerate closed interval.

Now we define $\mathscr{P}_{m+1}$. We start by setting $\mathscr{P}_m = \mathscr{P}_{m+1}$. Write $\mathscr{P}_{m+1} = \{I_1, \dots, I_l\}$ enumerated according to the orientation of $[0,1]$. Let $i \in \{1, \dots, l-1\}$ and let $a = g_m(I_i)$ and $b = g_m(I_{i+1})$. Let J be the smallest closed interval containing I_i and I_{i+1} and let $P = h_{m+1}(J \cap h_{m+1}^{-1}(T_{N,m+1}))$. There are two cases.

(1) If $|a - x|_\infty \leq 4(1/r)r^{m+1}$ for every $x \in P$, then we do not do anything. Note that in this case, $|a-b|_\infty \geq (\frac{1}{2}L_1\tau)r^{m+1}$ by Lemma 5.4(4) and the inductive hypothesis (P5).
(2) Suppose that the first case fails. Then, by (P5), there exists a component I' of $h_{m+1}^{-1}(V_N)$ such that I' is between I_i and I_{i+1} and such that both $|a - h_{m+1}(I')|_\infty$ and $|b - h_{m+1}(I')|_\infty$ are greater than r^{m+1}. In this case we add I' to $\mathscr{P}_{m+1}$.

We iterate the above procedure until only the first case applies for every two consecutive intervals of $\mathscr{P}_{m+1}$. This concludes the definition of $\mathscr{P}_{m+1}$.

Set $\mathscr{E}_{m+1}$ to be the collection of the components of $[0,1] \setminus \bigcup \mathscr{P}_{m+1}$ and define a continuous map $f_{m+1} : [0,1] \to \ell_\infty$ so that $f_{m+1}|I = h_{m+1}|I$ for all $I \in \mathscr{P}_{m+1}$ and $f_{m+1}|I$ is linear for all $I \in \mathscr{E}_{m+1}$.

We complete the inductive step by modifying h_{m+1} on each $I \in \mathscr{E}_{m+1}$, as in Step 1, which gives the definition of g_{m+1}. Towards this end, let $\{I_1, \dots, I_{2n_1}\}$ be an enumeration of $\mathscr{E}_{m+1}$. We start with I_1. If $I_1 = (x, y)$, then there exists a branch B of $T_{N,m+1+t}$ with respect to the arc $R_N(h_{m+1}(x), h_{m+1}(y))$, which is not contained in the image of h_{m+1} but contains a set K_w for some $w \in A^*(r^{m+1+t})$. We modify h_{m+1} on I_1 so that it traverses every edge of B linearly exactly twice, once in each direction and for every $z \in V_N \cap B$, every component of $g_{m+1}^{-1}(z)$ is nondegenerate. Again by Lemma 5.5, for each I_j, we can find such a branch B which is not contained in the image of $g_{m+1}|I_1 \cup \cdots I_{j-1}$ and we can define $g_{m+1}|I_j$ analogously.

Define now $\mathscr{F}_{m+1}$ to be the collection of components of $g_{m+1}^{-1}(V_N)$ and $\mathscr{B}_{m+1}$ be the collection of components of $[0,1] \setminus g_{m+1}^{-1}(V_N)$.

Properties (P1)–(P4) are immediate by design. Property (P5) follows from design of $\mathscr{P}_{m+1}$; see Case (1) in the construction of $\mathscr{P}_{m+1}$. Property (P6) follows from Lemma 5.5(3).

5.3.3. *Inductive step for $m+1 > N-t$.* The construction is as in the case $m+1 \leq N-t$. The only difference is that h_{m+1} is no longer modified. That is, $g_{m+1} = h_{m+1}$. Properties (P1)–(P5) are as in §5.3.2. Property (P6) is vacuous here.

Remark 5.6. Note that $f_N = g_N$, $\mathscr{P}_N = \mathscr{F}_N$ and $\mathscr{E}_N = \mathscr{B}_N$.

5.4. **Proof of Proposition 5.2.** By Lemma 3.2, we have that for all $m \in \{0, 1, \dots, N\}$ and all $w \in A^*(r^m)$,

$$\mathrm{card}(V_N \cap K_w) \simeq_{L_1,s,\tau} r^{-(N-m)s}.$$

Let $\psi : [0,1] \to [0,1]$ be the unique continuous and nondecreasing function such that

(1) $\psi(0) = 0$, $\psi(1) = 1$ and $\psi|I$ is constant for all $I \in \mathscr{P}_N$;
(2) $\psi|I$ is linear and $|\psi(I)| = (2\,\mathrm{card}(V_N) - 2)^{-1}$ for all $I \in \mathscr{E}_N$.

Using this function ψ, we can transform f_N into a $(1/s)$-Hölder map F_N.

Proof of Proposition 5.2. We first claim that there exists a unique continuous map $F_N : [0,1] \to T_N$ such that $f_N = F_N \circ \psi$. Indeed, the existence and uniqueness of F_N follow immediately from the fact that $f_N|I$ is constant for all intervals I such that $\psi(I)$ is a singleton, i.e., intervals in $\mathscr{P}_N$.

Next, we claim that $d_H(F_N([0,1]), K) \lesssim_{L_1,\tau} r^N$. If $x \in [0,1]$ and $y \in \psi^{-1}(x)$, then

$$\mathrm{dist}(F_N(x), K) = \mathrm{dist}(f_N(y), K) \le \mathrm{dist}(f_N(y), V_N) \le r^N.$$

On the other hand, if $x \in K$, then

$$\mathrm{dist}(x, F_N([0,1])) \le \mathrm{dist}(x, V_N) \le r^{N-1} \lesssim_{L_1,\tau} r^N.$$

It remains to show that F_N is $(1/s)$-Hölder with Hölder constant independent of N. For each $j \in \mathbb{N}$, define $F_N^j : [0,1] \to \ell_\infty$ so that $F_N^j \circ \psi = f_j$ with the convention $f_j = f_N$ if $j \ge N$. As with F_N, the maps F_N^j are well defined and unique. It is clear that as $j \to \infty$, F_N^j converges uniformly to F_N. The proof now is based on two claims.

We first claim that for all $j \in \mathbb{N}$ and all $x \in [0,1]$

$$|F_N^j(x) - F_N^{j+1}(x)|_\infty \lesssim_{L_1,\tau} r^j. \tag{5.3}$$

The claim is trivially true if $j \ge N$. Assume now that $j \in \{1, \dots, N-1\}$. Then, by design of the maps f_j,

$$|F_N^j(x) - F_N^{j+1}(x)|_\infty \le \mathrm{dist}_H(T_{N,j}, T_{N,j+1}) \lesssim_{L_1,\tau} r^j.$$

We now claim that for all $j \in \mathbb{N}$ and all $x, y \in [0,1]$

$$|F_N^j(x) - F_N^j(y)|_\infty \lesssim_{L_1,s,\tau,C_1,C_2} r^{j(1-s)}|x-y|. \tag{5.4}$$

If the claim is true for $j \le N$, then it is clearly true for all j. Therefore, for the proof of (5.4) we may assume that $j \le N$. We start by making two estimations. Fix $I \in \mathscr{E}_j$. Firstly, by (P5) we have that

$$\mathrm{diam}(F_N^j \circ \psi(I)) = \mathrm{diam}\, f_j(I) \le \mathrm{diam}\, g_j(I) \le (5/r) r^j \lesssim_{L_1,\tau} r^j.$$

If $j \le N - t$, then by (P6) there exists $w \in A^*(r^{j+t})$ such that $F_N(I)$ traverses all vertices of $K_w \cap V_N$. Therefore,

$$|\psi(I)| = \frac{\mathrm{card}\{J \in \mathscr{B}_N : J \subset I\}}{2\,\mathrm{card}(V_N) - 2} \ge \frac{\mathrm{card}(K_w \cap V_N)}{2\,\mathrm{card}(V_N) - 2}$$
$$\gtrsim_{L_1,s,\tau} \frac{r^{-(N-j-t)s}}{r^{-Ns}} \gtrsim_{L_1,s,\tau,C_1,C_2} r^{js}.$$

On the other hand, if $j > N - t$, then

$$|\psi(I)| \geq \frac{1}{2\,\mathrm{card}(V_N) - 2} \simeq_{L_1,s,\tau} r^{Ns} \gtrsim_{L_1,s,\tau,C_1,C_2} r^{js}.$$

Note that F_N^j is linear on each interval $\psi(I)$ with $I \in \mathscr{E}_j \cup \mathscr{P}_j$. Therefore,

$$\mathrm{Lip}(F_N^j) = \max_{I\in\mathscr{E}_j\cup\mathscr{P}_j} \frac{\mathrm{diam}(F_N^j(\psi(I)))}{|\psi(I)|} = \max_{I\in\mathscr{E}_j} \frac{\mathrm{diam}(F_N^j(\psi(I)))}{|\psi(I)|} \lesssim_{L_1,s,\tau,C_1,C_2} r^{j(1-s)}$$

and the second claim follows.

Now, by (5.3) and (5.4) and Lemma 2.7, the sequence $(F_N^j)_j$ converges uniformly to F_N and F_N is a $(1/s)$-Hölder map with Hölder constant depending only on L_1, s, τ, C_1, C_2. $\square$

6. Bedford-McMullen carpets and self-affine sponges

Self-affine carpets were introduced and studied independently by Bedford [Bed84] and Mcmullen [McM84]. Fix integers $2 \leq n_1 \leq n_2$. For each pair of indices $i \in \{1,\dots,n_1\}$ and $j \in \{1,\dots,n_2\}$, let $\phi_{i,j} : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine contraction given by

$$\phi_{i,j}(x,y) = (n_1^{-1}(i-1+x), n_2^{-1}(j-1+y)) \quad \text{with } \mathrm{Lip}\,\phi_{i,j} = n_1^{-1}.$$

For each nonempty set $A \subset \{1,\dots,n_1\}\times\{1,\dots,n_2\}$, we associate the iterated function system $\mathcal{F}_A = \{\phi_{i,j} : (i,j) \in A\}$ over $\mathbb{R}^2$ and let $\mathcal{S}_A$ denote the attractor of $\mathcal{F}_A$, called a *Bedford-McMullen carpet*. In general, we have $\mathcal{S}_A \subset [0,1]^2$.

The following proposition serves as a brief overview of how the similarity dimension of $\mathcal{F}_A$ compares to the Hausdorff, Minkowski, and Assouad dimensions of the carpet $\mathcal{S}_A$; for definitions of these dimensions, we refer the reader to [McM84] and [Mac11].

Proposition 6.1. *Let $2 \leq n_1 \leq n_2$ and A be as above. For all $i \in \{1,\dots,n_1\}$, define*

$$t_i := \mathrm{card}\{j : (i,j) \in A\}.$$

Also define $t := \max_i t_i$ and $r := \mathrm{card}\{i : t_i \neq 0\}$.

(1) *The similarity dimension is*

$$\text{s-dim}(\mathcal{F}_A) = \log_{n_1}\left(\sum_{i=1}^{n_1} t_i\right) = \log_{n_1}(\mathrm{card}\,A).$$

(2) [McM84] *The Hausdorff dimension is*

$$\dim_H(\mathcal{S}_A) = \log_{n_1}\left(\sum_{i=1}^{n_1} t_i^{\log_{n_2} n_1}\right).$$

(3) [McM84] *The Minkowski dimension is*

$$\dim_M(\mathcal{S}_A) = \log_{n_1} r + \log_{n_2}\left(r^{-1}\sum_{i=1}^{n_1} t_i\right) = \log_{n_1} r + \log_{n_2}(r^{-1}\,\mathrm{card}\,A).$$

(4) [Mac11] *If $n_1 < n_2$, then the Assouad dimension is*

$$\dim_A(\mathcal{S}_A) = \log_{n_1} r + \log_{n_2} t.$$

It is easy to see that $\dim_H(\mathcal{S}_A) \le \dim_M(\mathcal{S}_A) \le \min\{\dim_A(\mathcal{S}_A), \text{s-dim}(\mathcal{F}_A)\}$. However, no comparison can be made, in general, between $\dim_A(\mathcal{S}_A)$ and s-dim$(\mathcal{F}_A)$. We note that the similarity dimension of a carpet can exceed 2 (see Figure 2).

6.1. **Hölder parameterization of connected Bedford-McMullen carpets with sharp exponents.** For each index $i \in \{1, \dots, n_1\}$, define $A_i := \{i\} \times \{1, \dots, n_2\}$ and $A_0 := \bigcup_{i=1}^{n_1} A_i$. Note that the carpet $\mathcal{S}_{A_0} = [0,1]^2$, and for each $i \in \{1, \dots, n_1\}$, the carpet $\mathcal{S}_{A_i}$ is the vertical line segment $\{(i-1)/(n_1-1)\} \times [0,1]$ (see Figure 3).

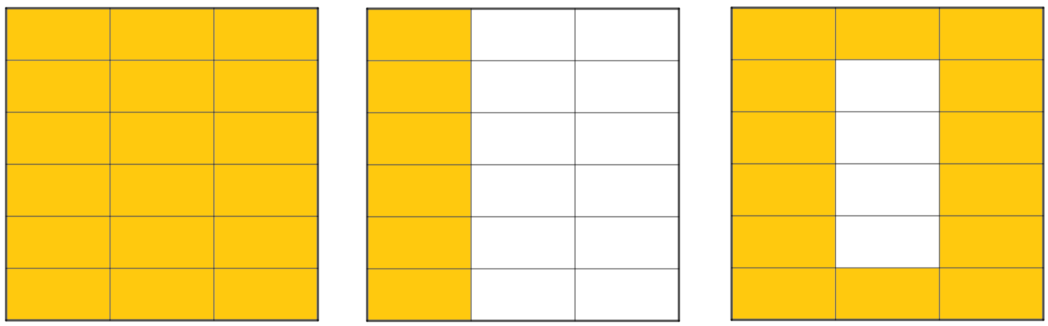

FIGURE 3. First iteration of Bedford-McMullen carpets generated by A. On the left, $A = A_0$ (the square). In the middle, $A = A_1$ (a vertical line). On the right, $A = \{(1,1), \dots, (1,6), (2,1), (2,6), (3,1), \dots, (3,6)\}$.

Our goal in this section is to establish the following statement, which encapsulates Theorem 1.4 from the introduction.

Theorem 6.2. *Let $2 \le n_1 \le n_2$ be integers and let A be as above. If $\mathcal{S}_A$ is connected, then there exists a surjective $(1/\alpha)$-Hölder map $F : [0,1] \to \mathcal{S}_A$ with*

$$\alpha = \begin{cases} 1, & \text{if } A = A_i \text{ for some } i \in \{1, \dots, n_1\}; \\ 2, & \text{if } A = A_0; \\ \text{s-dim}(\mathcal{F}_A), & \text{otherwise.} \end{cases}$$

Furthermore, the exponent α is sharp.

Note that the conclusion of Theorem 6.2 is trivial in the case that $A \in \{A_0, \dots, A_{n_1}\}$ or in the case that card $A = 1$. Below we give a proof of the sharpness of the exponent α, and in §6.2 we show why such a surjection exists.

Lemma 6.3. *If $\mathcal{S}_A$ is connected and $A \notin \{A_0, \dots, A_{n_1}\}$, then there exists a pair of indices $(i,j) \in A$ such that $j < n_2$ and $(i, j+1) \notin A$ or such that $j > 1$ and $(i, j-1) \notin A$.*

Proof. To establish the contrapositive, suppose that the conclusion of the lemma fails. Then $A = B \times \{1, \dots, n_2\}$ for some nonempty set $B \subset \{1, \dots, n_1\}$. If card$(B) = 1$, then $A = A_i$ for some $1 \le i \le n_1$. If $1 < \text{card}(B) < n_1$, then the carpet $\mathcal{S}_A$ is disconnected. Finally, if card$(B) = n_1$, then $A = A_0$. $\square$

Lemma 6.4. *Suppose that $\mathcal{S}_A$ is connected,* $\operatorname{card} A \geq 2$*, and $A \notin \{A_1, \dots, A_{n_1}\}$. Then the "first iteration" $\bigcup_{(i,j)\in A} \phi_{i,j}([0,1]^2)$ is a connected set that intersects both the left and the right edge of* $[0,1]^2$.

Proof. If $\operatorname{card} A \geq 2$, $A \notin \{A_1, \dots, A_{n_1}\}$, and the "first iteration" $\bigcup_{(i,j)\in A} \phi_{i,j}([0,1]^2)$ does not touch the left or right edge, then the "second iteration" $\bigcup_{(i,j),(k,l)\in A} \phi_{i,j} \circ \phi_{j,k}([0,1]^2)$ is disconnected. We leave the details as a useful exercise for the reader. It may help to visualize the diagrams in Figures 2 or 3. □

Corollary 6.5. *Suppose that $\mathcal{S}_A$ is connected,* $\operatorname{card}(A) \geq 2$*, and $A \notin \{A_1, \dots, A_{n_1}\}$. Then $\mathcal{S}_A$ intersects both left and right edge of* $[0,1]^2$.

We are ready to prove Theorem 6.2.

Proof of Theorem 6.2. With the conclusion being straightforward otherwise, let us assume that $\mathcal{S}_A$ is a connected Bedford-McMullen carpet with $\operatorname{card} A \geq 2$ and $A \notin \{A_0, \dots, A_{n_1}\}$. Let $s = \text{s-dim}\, \mathcal{F}_A$. We defer the proof of existence of a $(1/s)$-Hölder parameterization of $\mathcal{S}_A$ to §6.2, where we prove existence of Hölder parameterizations for self-affine sponges in $\mathbb{R}^N$ (see Corollary 6.7). It remains to prove the sharpness of the exponent s.

Set $k = \operatorname{card} A$ and suppose that $f : [0,1] \to \mathcal{S}_A$ is a $(1/\alpha)$-Hölder surjection for some exponent $\alpha > 0$. Since $\mathcal{S}_A$ has positive diameter, the Hölder constant $H := \text{Höld}_{1/\alpha} f > 0$. By Proposition 6.1, $\text{s-dim}\, \mathcal{F}_A = \log_{n_1}(k)$. Thus, we must show that $\alpha \geq \log_{n_1} k$.

Fix $m \in \mathbb{N}$ and let A^m, A^*, and ϕ_w be defined as in §3.1 relative to the alphabet $\{(i,j) : 1 \leq i \leq n_1, 1 \leq j \leq n_2\}$. For each $m \in \mathbb{N}$ and each word $w = (i_1, j_1) \cdots (i_m, j_m)$, set $S_w = \phi_w([0,1]^2)$. Let $(i_0, j_0) \in A$ be an index given by Lemma 6.3, i.e. an address in the first iterate such that the rectangle either immediately above or below is omitted from the carpet. Without loss of generality, we assume that $j_0 < n_2$ and $(i_0, j_0 + 1) \notin A$ (there is no rectangle below (i_0, j_0)). Moreover, we assume that

$$j_0 = \min\{j : (i_0, j) \in A \text{ and } (i_0, j + 1) \notin A\}.$$

For each word $w \in A^m$, we now define a "column of rectangles" $\tilde{S}_w$, as follows.

Case 1. If S_w intersects the bottom edge of $[0,1]^2$, then set $\tilde{S}_w = \bigcup_{j=0}^{j_0} S_{w(i_0,j)}$.

Case 2. Suppose that S_w does not intersect the bottom edge of $[0,1]^2$. Let $u = (i_1, j_1) \cdots (i_m, j_m)$ with

$$(i_1, j_1), \dots, (i_m, j_m) \in \{1, \dots, n_1\} \times \{1, \dots, n_2\}$$

such that the upper edge of S_u is the same as the lower edge of S_w. This case is divided into three subcases.

Case 2.1. Suppose that $u \notin A^m$. Then, as in Case 1, set $\tilde{S}_w = \bigcup_{j=0}^{j_0} S_{w(i_0,j)}$.

Case 2.2. Suppose that $u \in A$ and $u(i_0, n_2) \notin A^{m+1}$. Then we set $\tilde{S}_w = \bigcup_{j=0}^{j_0} S_{w(i_0,j)}$.

Case 2.3. Suppose that $u \in A^m$ and $u(i_0, n_2) \in A^{m+1}$. Let $j_1 = \max\{j : (i_0, j-1) \notin A\}$. Then we set $\tilde{S}_w = \left(\bigcup_{j=0}^{j_0} S_{w(i_0,j)}\right) \cup \left(\bigcup_{j=j_1}^{n_2} S_{u(i_0,j)}\right)$.

In each case, $\tilde{S}_w \cap \mathcal{S}_A$ is a connected set that intersects both the left and right edges of $\tilde{S}_w$, but does not intersect the rectangles S_u immediately above and below $\tilde{S}_w$. Moreover, the sets $\tilde{S}_w$ have mutually disjoint interiors. If τ_w is the line segment joining the midpoints of upper and lower edges of $\tilde{S}_w$, then τ_w contains a point of $\mathcal{S}_A$, which we denote by x_w.

Consequently, there exists $I_w \subset [0,1]$ such that $f(I_w)$ is a curve in $\tilde{S}_w$ joining x_w with one of the left/right edges of $\tilde{S}_w$. Clearly, the intervals I_w are mutually disjoint and

$$1 \geq \sum_{w\in A^m} \operatorname{diam} I_w \geq H^{-\alpha} \sum_{w\in A^m} (\operatorname{diam} f(I_w))^\alpha \gtrsim_{H,\alpha} \sum_{w\in A^m} (2n_1^{m+1})^{-\alpha} \gtrsim_{n_1,\alpha} (kn_1^{-\alpha})^m.$$

Since m is arbitrary, $\alpha \geq \log_{n_1} k$. □

6.2. **Lipschitz lifts and Hölder parameterization of connected self-affine sponges.** Analogues of the Bedford-McMullen carpets in higher dimensional Euclidean spaces are called *self-affine sponges*; for background and further references, see [KP96], [DS17], [FH17]. To describe a self-affine sponge, let $N \geq 2$ and let $2 \leq n_1 \leq \cdots \leq n_N$ be integers. For each n-tuple $\mathbf{i} = (i_1, \dots, i_N) \in \{1,\dots,n_1\} \times \cdots \times \{1,\dots,n_N\}$, we define an affine contraction $\phi_{\mathbf{i}} : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\phi_{\mathbf{i}}(x_1, \dots x_N) = (n_1^{-1}(i_1 - 1 + x_1), \dots, n_N^{-1}(i_N - 1 + x_N)) \quad \text{with } \operatorname{Lip}\phi_{\mathbf{i}} = n_1^{-1}.$$

For every nonempty set $A \subset \{1,\dots,n_1\} \times \cdots \times \{1,\dots,n_N\}$, we associate an iterated function system $\mathcal{F}_A = \{\phi_{\mathbf{i}} : \mathbf{i} \in A\}$ over $\mathbb{R}^N$ and let $\mathcal{S}_A$ denote the attractor of $\mathcal{F}_A$, which we call a self-affine sponge.

Our strategy to parameterize a connected Bedford-McMullen carpet or self-affine sponge is to construct a Lipschitz lift of the set to a self-similar set in a metric space for which we can invoke Theorem 1.3. Then the Hölder parameterization of the self-similar set descends to a Hölder parameterization of the carpet or sponge.

Lemma 6.6 (Lipschitz lifts). *Let $N \geq 2$ be an integer, let $2 \leq n_1 \leq \cdots \leq n_N$ be integers, and let A be a nonempty set as above. There exists a doubling metric d on $\mathbb{R}^N$ such that if $\widetilde{\mathcal{S}}_A$ denotes the attractor of the IFS $\widetilde{\mathcal{F}}_A = \{\phi_{\mathbf{i}} : \mathbf{i} \in A\}$ over $(\mathbb{R}^N, d)$, then*

(1) the identity map $\mathrm{Id} : \widetilde{\mathcal{S}}_A \to \mathcal{S}_A$ is a 1-Lipschitz homeomorphism;
(2) $\text{s-dim}\, \widetilde{\mathcal{F}}_A = \text{s-dim}\, \mathcal{F}_A = \log_{n_1}(\operatorname{card} A) =: s$, $\widetilde{\mathcal{S}}_A$ is self-similar, and $\mathcal{H}^s(\widetilde{\mathcal{S}}_A) > 0$.

Proof. Consider the product metric d on $\mathbb{R}^N$ given by

$$d((x_1,\dots,x_N),(x_1',\dots,x_N')) = \left(\sum_{i=1}^N |x_i - x_i'|^{2\log_{n_i} n_1}\right)^{1/2}.$$

In other words, d is a metric obtained by "snowflaking" the Euclidean metric separately in each coordinate. Note that if $n_1 = \cdots = n_N$, then d is the Euclidean metric. It is straightforward to check that $(\mathbb{R}^N, d)$ is a doubling metric space and the identity map $\mathrm{Id} : (\mathcal{S}_A, d) \to \mathcal{S}_A$ is a 1-Lipschitz homeomorphism; e.g. see Heinonen [Hei01]. We now claim that the affine contractions $\phi_{\mathbf{i}}$ generating the sponge $\mathcal{S}_A$ become similarities in the metric space $(\mathbb{R}^N, d)$. Indeed, let $\mathbf{i} = (i_1, \dots, i_N) \in A$. Then

$$\begin{aligned} d(\phi_{\mathbf{i}}(x_1, \dots, x_N), \phi_{\mathbf{i}}(x_1', \dots, x_N')) &= \left(\sum_{i=1}^{N} n_i^{-2\log_{n_i} n_1} |x_i - x_i'|^{2\log_{n_i} n_1} \right)^{1/2} \\ &= n_1^{-1} d((x_1, \dots, x_N), (x_1', \dots, x_N')). \end{aligned}$$

Since each of the similarities $\phi_{\mathbf{i}}$ have scaling factor n_1^{-1}, it follows that

$$\text{s-dim}(\widetilde{\mathcal{F}}_A) = \text{s-dim}\, \mathcal{F}_A = \log_{n_1}(\operatorname{card} A) =: s$$

Finally, $\widetilde{\mathcal{F}}_A$ satisfies the strong open set condition (SOSC) with $U = (0,1)^N$. Therefore, $\mathcal{H}^s(\widetilde{\mathcal{S}}_A) > 0$ by Theorem 2.3, since doubling metric spaces are β-spaces. □

Corollary 6.7. *If $\mathcal{S}_A$ is a connected self-affine sponge in $\mathbb{R}^N$, then $\mathcal{S}_A$ is a $(1/s)$-Hölder curve, where $s = \log_{n_1}(\operatorname{card} A)$ is the similarity dimension of $\mathcal{F}_A$.*

Proof. Let $\widetilde{\mathcal{S}}_A$ denote the lift of the sponge $\mathcal{S}_A$ in Euclidean space $\mathbb{R}^N$ to the metric space $(\mathbb{R}^N, d)$ given by Lemma 6.6. By Lemma 6.6 (2), the lifted sponge $\widetilde{\mathcal{S}}_A$ is a self-similar set and $\mathcal{H}^s(\widetilde{\mathcal{S}}_A) > 0$, where $s = \text{s-dim}\, \widetilde{\mathcal{F}}_A = \text{s-dim}\, \mathcal{F}_A = \log_{n_1}(\operatorname{card} A)$. By Remes' theorem in metric spaces (Theorem 1.3), there exists a $(1/s)$-Hölder surjection $F : [0,1] \to \widetilde{\mathcal{S}}_A$. By Lemma 6.6 (1), the identity map $\mathrm{Id} : \widetilde{\mathcal{S}}_A \to \mathcal{S}_A$ is a Lipschitz homeomorphism. Therefore, the composition $G = [0,1] \to \mathcal{S}_A$, $G := \mathrm{Id} \circ F$ is a $(1/s)$-Hölder surjection. □

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