Updates on Traveling Salesman in Banach Spaces

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X metric space
\[ \Gamma \subset X \] is a \textbf{cone} if \[ \Gamma = f([0,1]) \]

for some cts map \( f : [0,1] \rightarrow X \)

\( f \) is a \textbf{parameterization} of \( \Gamma \)

\[
\text{var}(f) = \sup \left\{ \sum_{i} |f(x_i) - f(x_{i-1})| : \text{Partitions of } [0,1] \right\}
\]

"Intrinsic Length"
\[ \text{var}(f) = \sup \left\{ \sum_i |f(x_i) - f(x_{i-1})| : \text{Partitions of } [0,1] \right\} \]

"Intrinsic Length"

\[ H'(\Gamma) = \lim_{\delta \to 0} \inf \left\{ \sum_i \text{diam } U_i : \Gamma \subset \bigcup_i U_i \text{ and } \text{diam } U_i \leq \delta \right\} \]

"Extrinsic Length"

1-dimensional Hausdorff Measure
Ważewski’s Theorem

\( X \) is a metric space
\( \Gamma \subset X \) nonempty

1. \( \Gamma \) is a **rectifiable curve**, i.e. \( \Gamma = f([0,1]) \) for some \( f \) with \( \text{var}(f) < \infty \)

2. \( \Gamma \) is compact, connected, and \( \mathcal{H}^1(\Gamma) < \infty \)

3. \( \Gamma \) is a **Lipschitz curve**, \( \Gamma = f([0,1]) \) for some \( f \) s.t. \( |f(x) - f(y)| \leq L |x-y| \)
\[ l_p = \left\{ (x_1, x_2, x_3, \ldots) \in \mathbb{R}^\omega : \sum_i |x_i|^p < \infty \right\} \]

- Banach space when \( 1 \leq p < \infty \)
  \[ |x|_p = \left( \sum_i |x_i|^p \right)^{\frac{1}{p}} \]
  \( \text{dist}_p(x, y) = |x-y|_p \)

- Separable (\( 1 \leq p < \infty \)), reflexive (\( 1 < p < \infty \))

- Increasing:
  \[ p < q \implies l_p \subset l_q \]
  Identily is \( 1 \)-Lipschitz embedding:
  \[ |x|_q \leq |x|_p \]

Corollary: \( \Gamma \) rectifiable in \( l_p \) \( \implies \) \( \Gamma \) rectifiable in \( l_q \)
\[ H'(Γ) = \| (1,1) - (0,0) \|_p \]
\[ = 2^{1/p} \]
- rectifiable in each \( l_p \)
- shorter as \( p \to ∞ \)

- In finite-dimensions, rectifiability independent of norm
- but length of curve depends on norm
- What about in infinite-dimensions?

In \( l_1 \), there are infinitely many geodesics between \((0,0)\) and \((1,1)\)
Example (B-McCordy) Related Example by Edefon-Naber-Valtorta

For all \(1 < p < \infty\), there exists a curve \(\Gamma\) in \(\ell_p\) s.t.
\[\chi_{\ell_p}(\Gamma) = \infty\] and \[\chi_{\ell_q}(\Gamma) < \infty\] for all \(q < p\).

Add blips of relative height \(l_2\) in \(e_3\) direction.

Add blips of relative height \(l_3\) in \(e_4\) direction ...
Example (B-McCordy) Related Example by Boden-Ahler-Vanderkya

For all $1 < p < \infty$, there exists a curve $\Gamma$ in $L^p$ s.t.

$\mathcal{H}_p^1(\Gamma) = \infty$ and $\mathcal{H}_q^p(\Gamma) < \infty$ for all $q > p$.

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$\Gamma \subset L^p$ relative heights $\gamma_i$

$\mathcal{H}_p^1(\Gamma) \approx \exp\left(\sum \gamma_i^p\right)$

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Choose $\gamma_i = \frac{\delta}{i \log(i + i_0)}$ with $\delta > 0$, $i_0 \geq 1$ so $\gamma_i \leq \frac{1}{i^2}$.

$\mathcal{H}_p^1(\Gamma) = \infty$, but $\mathcal{H}_q^p(\Gamma) \leq \exp\left(\sum \frac{\delta^{q/p}}{i \log(i + i_0)^{q/p}}\right) < \infty$

when $p < q$. 

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Rectifiable $\iff \sum \gamma_i^p < \infty$

If rectifiable, then $\Gamma$ is Ahlfors regular.
We still do not have a complete picture!

\[ \ell_p^2 = \left( \mathbb{R}^2, 1 \cdot \ell_p \right) \]

von Koch curve
- add blips in "I" directions
- relative heights \( \gamma_i \)

\[ \mu'_{\ell_p}(\Gamma) \approx \mu'_{e_2}(\Gamma) \approx \exp(\sum \gamma_i^2) \]

Length gained by adding blips sensitive to direction of blip!

\[ \ell_p \text{ infinite-dimensions} \]

von Koch curve
- add blips in new \( e_i \), directions
- relative heights "\( \gamma_i \)"

\[ \mu'_{\ell_p}(\Gamma) \approx \exp(\sum \gamma_i^p) \]

Question: Can you build \( \Gamma, \mu'_{\ell_p}(\Gamma) \approx \exp(\sum \gamma_i^2), \gamma \in [2, p] \)?
Analyst's Traveling Salesman Problem

P. Jones (1990): Given a set $E$ in a metric space $X$, decide whether or not $E$ is contained in some rectifiable curve $\Gamma$. If so, find a curve $\Gamma \supset E$ “short as possible.”

Full solutions for sets in

$\mathbb{R}^2$ (P. Jones, 1990)

$\ell^2$ (R. Schul, 2007)

Radon measures in $\mathbb{R}^n$ (M.B., R. Schul 2017)

$\mathbb{R}^n$ (K. Okikiolu, 1992)

Carnot Groups (S. Li 2019)

Graph Inverse Limit Spaces (G.C. David, R. Schul 2017)
Partial Survey (Continued)

I. Hahlomaa (2005)

- Sufficient Conditions for \( \exists \Gamma \supseteq E \)
in arbitrary metric space \( X \)
- Condition is not necessary in \( l^2_1 = (\mathbb{R}^2, 1-1) \)

G.C. David, R. Schul (2019)

- Necessary Conditions for \( \Gamma \) to be rectifiable
  in arbitrary metric space \( X \) when \( \Gamma \) doubling

N. Edelen, A. Naber, D. Valtorta (2019)

- Sufficient Conditions for \( \exists \) bi-Lipschitz surface \( \supseteq E \)
in \( l^2 \) and for \( \exists \) bi-Lip curve \( \supseteq E \) in \( l^p, 1 < p < \infty \)
P. Jones $\beta$ number in a Banach space

"unilateral linear approximation"

Set $E$

Window $\Omega$

Line $L$

\[
\beta_E(\Omega, L) = \sup_{x \in E \cap \Omega} \frac{\text{dist}(x, L)}{\text{diam} \Omega} \in [0, 1]
\]

\[
\beta_E(\Omega) = \inf_L \beta_E(\Omega, L)
\]
Jones-Okikiolu Theorem in Banach spaces

$(X,1,1)$ finite-dimensional Banach space

$\Delta$ system of dyadic cubes (choice of basis)

$E \subset X$ bounded set

$\exists \Gamma \supseteq E$ with $\mathcal{H}'(\Gamma) < \infty$ iff

$$S_E = \sum_{\Omega \in \Delta} \beta_E (3\Omega)^2 \cdot \text{diam } \Omega < \infty$$

Moreover, can find $\Gamma$ with $\mathcal{H}'(\Gamma) \asymp \text{diam } E + S_E$

where implicit constants only depend on $\dim X$, $\Delta$ (choice of basis) and norm 1,1
Challenges in Infinite-Dimensions

1. No "Dyadic Cubes"
   - Many Good Ideas by R. Schul

   - Solution: Use $2^{-k}$-nets $X_k$ for $E$
     and multi-resolution families $\{B(x,3 \cdot 2^{-k})\}_{x \in X_k}$

2. Uncontrolled Overlap
   - If $E = \Gamma$ and $\Psi'(\Gamma) < \infty$, $X_k$ locally finite but
   - can be arbitrarily large number of balls $B(y,3 \cdot 2^{-k})$
     that intersect $B(x,3 \cdot 2^{-k})$

   - Soln: Complicated, but use fact when this happens $\beta$ large
Theorem (R. Schul 2007) \[ EC_{l^2} \text{ bounded is contained in rectifiable curve} \]

iff \[ \sum_{Q \in G} B_E(Q)^2 \text{diam } E < \infty \]

\[ \uparrow \text{ Multi-resolution Family for } E \]

Theorem (B-McCandlly 2020/2021) \[ 1 < p < \infty \]

\[ EC_{l^p} \text{ bounded} \]

- If \[ \sum_{Q \in G} B_E(Q) \cdot \text{diam } Q < \infty \], then \[ EC \Gamma_{N_1} \cdot \Gamma < \infty \]

- If \[ EC \Gamma \cdot N_1 < \infty \], then \[ \sum_{Q \in G} B_E(Q)^\min(p,2) \cdot \text{diam } Q < \infty \]

- If \[ EC \Gamma \cdot N_1 < \infty \], then \[ \sum_{Q \in G} B_E(Q)^\max(p,2) \cdot \text{diam } Q < \infty \]

Examples show gap btw min(p,2) and max(p,2) cannot be filled in.

\[ \uparrow \text{ Modulus of Smoothness} \]

\[ \uparrow \text{ Modulus of Convexity} \]
Takeaways

1. Analyst's TSP
   Trying to understand what rectifiable curves and their subsets look like

2. Still Open!
   We only have solutions in a few metric spaces
   Euclidean/Carathéodory methods not strong enough

3. Length gain is sensitive to direction
   In spaces like $\ell_p, p \neq 2$, we don't understand how to effectively estimate length gain
   Beta numbers are not strong enough